Categorical Duality in Probability and Quantum Foundations

Robert Furber

September 19, 2016

Contents

Introduction

When we consider the semantics of programs, we can consider *state transformers* and *predicate transformers*. A *state transformer* describes the action of the program, taking its initial state to its final state. Predicate transformers go in the opposite direction and describe how to take a predicate to its *weakest precondition*. In the classical case of nondeterminism [104, 124] an isomorphism between state transformers and predicate transformers is obtained by restricting predicate transformers to those that are *healthy*[124].

This is the reason for our focus on categorical dualities, as the relationship between state and predicate transformers is best expressed as a contravariant equivalence of categories.

In the first chapter, we concern ourselves with probabilistic computation. We prove that $\mathcal{K}\ell(\mathcal{R}) \simeq \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$, which can be considered to be a probabilistic generalization of Gelfand duality. We also give a brief introduction to effect modules, their relationship to order-unit spaces, and the expectation monad.

Chapters two to four can all be considered as a non-commutative generalization of chapter one. In the second chapter, we introduce base-norm spaces and their relationship to convex sets. This is mainly for application in the next chapter. Base-norm spaces are intended as a way of "freely" producing a vector space for a convex set to live inside. The chapter fills a gap in the literature, as although the definition of order-unit space is generally agreed upon, each author has their own definition of base-norm space, and they do not compare their definitions to the definitions of other authors with explicit examples, which we do here. We also give characterizations of which convex sets correspond to which kind of base-norm space in two cases (we call them **BConv** and **CBConv**). We then give an adjunction between base-norm and order-unit spaces, based on the duality between states and effects, and restrict this to an equivalence. Then we briefly discuss how this equivalence is not quite adequate as a generalization of chapter one.

In the third chapter, we first introduce a new characterization of Akbarov's *Smith spaces* and their duality with Banach spaces. From this we can produce two dualities involving base-norm and order-unit spaces, depending on whether one takes the base-norm spaces or the order-unit spaces to be Smith spaces. If we take the base-norm spaces to be Smith, we get a state-and-effect triangle for C*-algebras, which gives us one possible state-transformer and predicate-transformer pair of semantics for quantum programs. In fact, the state-transformer semantics corresponds to what is called the *Schrödinger picture*, and the predicate transformer semantics to the *Heisenberg picture*.

We also produce another state-and-effect triangle, involving W*-algebras and normal maps, where the order-unit spaces are Smith. We show in each case (whether it is C* or W*-algebras) how to turn the base-norm and order-unit space equivalences into equivalences between a category of convex sets (**CBConv** or **CCL**) and a category of effect modules (**CEMod** and **BEMod**, respectively). We would prefer, in fact, to prove the dualities directly in this setting, but we could only do so using the extra facilities available in the vector space setting (linear independence, locally convex topologies, the Hahn-Banach and bipolar theorems).

These dualities give two generalizations of the duality between states and effects in [27, §3.4] to the infinite-dimensional case, although with the drawback that we consider only positive maps and not completely positive ones. A similar generalization was considered before by Rennela [110, Theorem 4.1, Appendix C] (see also [20, Proposition 5.1]). Rennela's version was more order-theoretic, using a different characterization of normal maps of W*-algebras, and with an adjunction between states and effects in the general case. This adjunction is not known to be an equivalence. This is a difficulty we were able to circumvent using locally convex topologies.

In the third chaper we also characterize **CBConv** as a reflective subcategory of a category of Eilenberg-Moore algebras, $\mathcal{EM}(\mathcal{D})$. In chapter four we use a theorem of Świrszcz to show that **CCL** has two characterizations, as $\mathcal{EM}(\mathcal{R})$ and $\mathcal{EM}(\mathcal{E})$ (algebras of the Radon and expectation monads, respectively). We can then characterize **CEMod**, the compact effect modules, independently of an embedding in a topological vector space.

Finally, we come to chapter five. We saw in chapter three that we can work either with C*-algebras or W*-algebras. In chapter one, however, we worked only with C*-algebras in the commutative case. It is known that commutative unital C*-algebras correspond to compact Hausdorff spaces (Gelfand duality), and the functor taking a compact Hausdorff space X to a commutative C*-algebra takes the algebra of continuous $\mathbb{C}$-valued functions under pointwise multiplication, *i.e.* $C(X)$. It is folklore that commutative W*-algebras correspond to measure spaces, up to equivalence of measures, where given a measure space (X, Σ, μ) one takes the algebra of bounded $\mathbb{C}$-valued measurable functions, modulo null functions[1], *i.e.* $L^\infty(X, \Sigma, \mu)$. However, to get an equivalence of categories like Gelfand duality, and even to get a W*-algebra from $L^\infty(X, \Sigma, \mu)$, many conditions are needed. In particular, it appears that normal maps (maps that produce normal morphisms of W*-algebras) have never previously been characterized, so we give what we believe is a new defintion thereof in this chapter. It is then possible to define a category $\mathcal{M}eas$ of measure spaces such that L^∞ defines an equivalence $\mathcal{M}eas \simeq \mathbf{CW^*Alg}^{\mathrm{op}}$. We have to leave it to future work to produce the analogous result to $\mathcal{K}\ell(\mathcal{R}) \simeq \mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}}$, as we do not have the space or time to do so here. We do verify that the infinite

[1]Functions supported on a set of measure zero.

distribution monad's Kleisli category $\mathcal{K}\ell(\mathcal{D}_\infty)$, does not provide the answer.

We now outline the original contributions. The probabilistic Gelfand duality in the first chapter is new. In the seconod chapter, the characterization of bases of base-norm spaces as sequentially complete bounded convex sets is new, as is the adjunction and equivalence for base-norm and order-unit spaces, in this categorical form. In chapter 3, the generalization of Akbarov's characterization of Smith spaces is new, and Smith base-norm and order-unit spaces are new definitions, so the equivalences given there are new. The universal enveloping compact effect module described there is also new. In chapter four, the intrinsic definition of a compact effect module is new. In chapter five, the definition of a normal map of measure spaces is new, as is the complete proof of the categorical equivalence between measure spaces and commutative W*-algebras, as only parts of it or restricted cases had previously appeared.

Acknowledgements

This research has been financially supported by the Netherlands Organisation for Scientific Research (NWO) under TOP-GO grant no. 613.001.013 (The logic of composite quantum systems).

The author would like to thank Kenta Cho, Tobias Fritz, David Handelman, Chris Heunen, Pieter Hofstra, Bart Jacobs, Klaas Landsman, Ieke Moerdijk, Prakash Panangaden, Mathys Rennela, Phil Scott, Alex Simpson, Sander Uijlen, Bas Westerbaan and Bram Westerbaan for helpful conversations and insights.

Chapter 0

Preliminaries

The section on monads originated in [64]

There are a number of preliminaries we must get out of the way.

0.1 Convexity in Vector Spaces

Recall that a topological vector space is a vector space equipped with a topology such that addition and scalar multiplication are continuous with respect to the topology of the field over which the space is defined (in our case, $\mathbb{R}$ or $\mathbb{C}$)[118, I.1]

A subset S of a (real) vector space E is *convex* if for each $x, y \in S$ and $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in S$. An equivalent characterization is that for any finite sequence $(\alpha_i)_{i \in I}$ of elements of $[0,1]$ such that $\sum_{i \in I} \alpha_i = 1$, and $(x_i)_{i \in I}$ a finite sequence of elements of S, we have $\sum_{i \in I} \alpha_i x_i \in S$. The intersection of a family of convex subsets is convex, so convex sets form a lattice. The smallest convex set containing a subset S of a vector space E is called the *convex hull* and is written $\mathrm{co}(S)$. It can be equivalently defined as the set of convex combinations of elements of S. A subset of a topological vector space is *σ-convex* if the analogous property for countable sequences $(\alpha_i)_{i \in I}$ and $(x_i)_{i \in I}$ holds, where the sum is interpreted as a limit of the sequence of finite sums in the usual manner.

We say a subset X of a vector space E is *absolutely convex* if for each finite set $x_1, \ldots, x_n$ in X and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\sum_{i=1}^n |\alpha_i| \leq 1$ then $\sum_{i=1}^n \alpha_i x_i \in X$. The definition resembles that of convexity but with absolute values (and ≤ 1 instead of $= 1$). We include the use of the empty set when forming convex combinations, so every absolutely convex set contains 0. The effect of this is to rule out $\emptyset$ as an absolutely convex subset of E, in spite of the fact that it is usually considered a convex subset of E. The astute reader will notice when this definition is required later on. There is a monadic theory of absolutely convex sets analogous to that for convex sets [107, 108], but as we do not require absolutely convex sets as independent objects we do not use it.

Unlike convexity, absolute convexity does not require an ordered field for definition, and so can also be defined for the complex numbers or even more general fields with valuation, such as p-adic fields. However, we will mostly use the real definition. For non-empty sets, absolute convexity is variously known as being *balanced and convex*[24, p. 102] or *circled and convex*[118, Chapter II, Exercise 1]. This is because a subset S of a real vector space E is *balanced* if $x \in S$ implies $-x \in S$, and being balanced and convex is the same as being absolutely convex (Lemma A.3.1 in the appendix). We use the notation $\mathrm{absco}(X)$ to refer to the absolutely convex hull of $X \subseteq E$, E a (real) vector space.

Lemma 0.1.1. *Let C be a non-empty subset of a vector space E. The absolutely convex hull of C is $\mathrm{co}(C \cup -C)$.*

Proof.

- $\mathrm{co}(C \cup -C) \subseteq \mathrm{absco}(C)$:

  Suppose we have some element of $x \in \mathrm{co}(C \cup -C)$, written as a convex combination

$$x = \alpha_1 x_1, \ldots, \alpha_k x_k + \alpha_{k+1}(-x_{k+1}) + \ldots + \alpha_n(-x_n)$$

  where $1 \leq k \leq n$, possibly reordering so elements of X occur for $i < k$ and elements of $-X$ for $i \geq k$. We can define $\{\beta_i\}_{1 \leq i \leq n}$ by taking $\beta_i = \alpha_i$ for $1 \leq i < k$ and $\beta_i = -\alpha_i$ for $i \geq k$. We have that $|\beta_i| = \alpha_i$ and so

$$\sum_{i=1}^{n} |\beta_i| = \sum_{i=1}^{n} \alpha_i = 1,$$

  which means that x can also be expressed as the absolutely convex combination

$$x = \sum_{i=1}^{n} \beta_i x_i,$$

  showing $x \in \mathrm{absco}(C)$.

- $\mathrm{absco}(C) \subseteq \mathrm{co}(C \cup -C)$:

  Suppose we have $x \in \mathrm{absco}(C)$, expressed as an absolutely convex combination

$$x = \sum_{i=1}^{n} \alpha_i x_i,$$

  allowing $n = 0$. We define $\beta_i = |\alpha_i|$ and $y_i = \mathrm{sgn}(\alpha_i) x_i$. There is still the problem that $\sum_{i=1}^{n} \beta_i$ could be strictly less than 1. We can pick an element of $y \in C$, as it is non-empty, and define $y_{n+1} = y$ and $y_{n+2} = -y$, and

$$\beta_{n+1} = \beta_{n+2} = \frac{1 - \sum_{i=1}^{n} \beta_i}{2}.$$

If $n = 0$, these are the only two values of β_i that are defined.

Then the convex combination

$$\sum_{i=1}^{n+2} \beta_i y_i = \sum_{i=1}^{n} |\alpha_i| \operatorname{sgn}(\alpha_i) x_i + \beta_{n+1} y - \beta_{n+2} y$$
$$= \sum_{i=1}^{n} \alpha_i x_i + \beta_{n+1} y - \beta_{n+1} y = x,$$

which proves that $x \in \operatorname{co}(C \cup -C)$. □

We say a subset S of a vector space E is *radially bounded* if for each line L through the origin, $S \cap L$ is bounded in L. Boundedness in L is defined as boundedness in $\mathbb{R}$ via any linear isomorphism $L \cong \mathbb{R}$. We say S is *radially compact* if it is radially bounded and $S \cap L$ is always closed in L, or equivalently by the Heine-Borel theorem, that $S \cap L$ is compact.

Lemma 0.1.2. *An absolutely convex set U is radially bounded iff it contains no line through the origin.*

Proof. If L is a line through the origin contained in U, $L \cap U = L$ and is therefore unbounded, so U is not radially bounded.

For the other way, suppose U is radially unbounded, which is to say that there exists a line L such that $L \cap U$ is unbounded. This means that under an isomorphism of L with $\mathbb{R}$, for each $n \in \mathbb{N}$ there is an $x \in \mathbb{R}$ such that $|x| \geq n$. Given such an x, by absolute convexity of U and L, we have $-x \in L \cap U$ also, and by Lemma 0.1.1 $[-n, n] \subseteq \operatorname{co}(\{-x\} \cup \{x\}) = [-x, x] \subseteq L \cap U$. Therefore the union of these intervals, the whole of L, is contained in U. □

The main reason for considering absolutely convex sets is their relationship to seminorms. We first define a relation between subsets of a real vector space. We say U *absorbs* V, for $U, V \subseteq E$, E a vector space, if there is some nonnegative real α such that $V \subseteq \alpha U$. Given a set $U \subseteq E$, we say it is *absorbent*[1] if for all $x \in E$, there is some λ such that $x \in \lambda U$, equivalently if $\lambda^{-1} x \in U$. This is the same as saying that U absorbs all singletons.

Lemma 0.1.3. *In a real vector space E, with subsets S, T:*

(i) If S and T are absorbent, then $S \cap T$ is absorbent. Therefore absorbent sets are closed under finite intersection.

(ii) If $S \subseteq T$ and S is absorbent, T is absorbent.

Taken together, these show that absorbent sets are a filter on E.

Proof.

[1] Known as *radial* in [118].

(i) Let $x \in E$. There exist $\alpha, \beta \in \mathbb{R}_{>0}$ such that $x \in \alpha S$ and $x \in \beta T$. Take $\gamma = \max\{\alpha, \beta\}$, and observe that $x \in \gamma S$ and $x \in \gamma T$, so $x \in \gamma S \cap \gamma T$. Since $\gamma > 0$, $\gamma \cdot \text{-}$ is a bijection, so $\gamma \cdot \text{-}$ preserves Boolean operations, and $\gamma S \cap \gamma T = \gamma(S \cap T)$, so $S \cap T$ is absorbent.

(ii) Let $x \in E$. There exists $\alpha \in \mathbb{R}_{>0}$ such that $x \in \alpha T \subseteq \alpha S$, so S is absorbent. □

Lemma 0.1.4. *In a topological vector space, every neighbourhood of* 0 *is absorbent.*

Proof. Let E be a topological vector space, $N \subseteq E$ a neighbourhood of 0, and $x \in E$. As scalar multiplication is continuous, $\text{-} \cdot x : \mathbb{R} \to E$ is continuous. Therefore there is an $\epsilon > 0$ such that $(\epsilon, \epsilon) \cdot x \subseteq N$. We therefore have that $\frac{\epsilon}{2}x \in N$, and so $x \in \frac{2}{\epsilon}N$. □

For absorbent absolutely convex sets, we can define the *Minkowski functional* (or *gauge*[118, II.1.4, page 39]), as

$$\|x\|_U = \inf\{\lambda > 0 \mid x \in \lambda U\}.$$

This defines a seminorm for each absorbent absolutely convex set U.

Lemma 0.1.5. *The following are equivalent for an absorbent absolutely convex subset U of E:*

(i) $\|\text{-}\|_U$ *is a norm.*

(ii) The only linear subspace of U is $\{0\}$.

(iii) U contains no line through the origin.

(iv) U is radially bounded.

Proof.

- (i) ⇒ (ii): Let $F \subseteq E$ be a linear subspace such that $F \subseteq U$. Then for all $\alpha \in [0, \infty)$, $F = \alpha F \subseteq \alpha U$, so we have that for all $x \in F$, $\|x\|_U = 0$ and so $x = 0$ by $\| - \|_U$ being a norm.

- (ii) ⇒ (iii): A line through the origin is a linear subspace not equal to $\{0\}$.

- (iii) ⇒ (i): We prove the contrapositive. Suppose $\|\text{-}\|_U$ is not a norm, *i.e.* that there exists $x \in E$, $x \neq 0$ such that $\|x\| = 0$. For any $\alpha > 0$, we have that $x \in \alpha^{-1}U$ and so $\alpha x \in U$. By absolute convexity, $\alpha x \in U$ for negative and zero values too, and so the line generated by x lies in U, and it is non-trivial because $x \neq 0$.

- (iii) ⇒ (iv): See Lemma 0.1.2. □

In fact, the Minkowski functional and open unit ball define isomorphisms between open absolutely convex neighbourhoods of 0 and continuous seminorms in any topological vector space [118, II.1.5 and 1.6].

If $(E, \|\text{-}\|)$ is a seminormed space, we define $\mathrm{Ball}(E)$, or $\mathrm{Ball}(\|\text{-}\|)$ to disambiguate if there is more than one seminorm present, as

$$\mathrm{Ball}(E) = \{x \in E \mid \|x\| \leq 1\}.$$

In the case that the seminorm is defined as the Minkowski functional of some set, we can show the following.

Lemma 0.1.6. *Let E be a real vector space and $U \subseteq E$ be an absolutely convex absorbent set. Then $\|x\|_U \leq 1$ iff $x \in \alpha U$ for all α such that $1 < \alpha < \infty$. Equivalently*

$$\mathrm{Ball}(E) = \bigcap_{1<\alpha<\infty} \alpha U.$$

Proof. Let $x \in E$.

- $\|x\|_U \leq 1 \Rightarrow \forall \lambda > 1. x \in \lambda U$:

  If $\|x\|_U \leq 1$, this means that $\inf\{\lambda > 0 \mid x \in \lambda U\} \leq 1$. If $\lambda > 1$, then $\inf\{\lambda > 0 \mid x \in \lambda U\} < \lambda$, so λ is not a lower bound for $\{\lambda' > 0 \mid x \in \lambda' U\}$, so there exists some $\lambda' > 0$ such that $x \in \lambda' U$ and $\lambda \not\leq \lambda'$, *i.e.* $\lambda > \lambda'$. Therefore $\lambda' U \subseteq \lambda U$, so $x \in \lambda U$.

- $(\forall \lambda > 1. x \in \lambda U) \Rightarrow \|x\|_U$:

  Suppose that $x \in \lambda U$ for all $\lambda > 1$. Then we have

  $$(1, \infty) \subseteq \{\lambda > 0 \mid x \in \lambda U\},$$

  so

  $$\|x\| = \inf\{\lambda > 0 \mid x \in \lambda U\} \leq \inf(1, \infty) = 1. \qquad \square$$

Recall that a norm $\|\text{-}\|$ on a vector space E defines a metric $d(x, y) = \|x - y\|$, and this metric defines a topology on E, the $\|\text{-}\|$-topology[118, II.2][24, III.1]. A Banach space is a normed space in which this metric is complete.

Lemma 0.1.7. *If E is a real vector space and $U \subseteq E$ a radially compact absolutely convex absorbent set, the closed unit ball of $\|\text{-}\|_U$ is U.*

Proof. The closed unit ball is $U' = \{x \in E \mid \forall \lambda \in (1, \infty). x \in \lambda U\}$ by Lemma 0.1.6. If $x \in U$, and λ is a real number > 1, then $\lambda^{-1} \in (0, 1)$, so $\lambda^{-1} \cdot x \in U$. Therefore $\lambda\lambda^{-1} \cdot x \in \lambda U$ so $x \in \lambda U$, and hence $x \in U'$. This means $U \subseteq U'$.

Now suppose $x \in U'$. If $x = 0$, then $x \in U$, so we reduce to the case $x \neq 0$. Let L be the line generated by x. Consider the set $M = \{\alpha^{-1} x \mid \alpha > 1\}$. Since $x \in U'$, we have that for all $\alpha > 1$, $\alpha^{-1} x \in U$, and therefore $M \subseteq U$. By linearity, $M \subseteq U \cap L$. As E is a base-norm space, $U \cap L$ is compact, and therefore closed, so the closure of M is also contained in $U \cap L$. Therefore we only need to show that $x \in \overline{M}$. We do this by showing that every neighbourhood of x intersects M.

Let $\epsilon > 0$, define $\epsilon' = \max\{\epsilon, \frac{3}{2}\}$ and define

$$\alpha = \frac{1}{1 - \frac{\epsilon'}{2}}.$$

Since $0 < 1 - \frac{\epsilon'}{2} < 1$, we have $\alpha > 1$, as well as being defined. Now

$$\begin{aligned} \|x - \alpha^{-1}x\|_U &= \|(1-\alpha^{-1})x\|_U = (1-\alpha^{-1})\|x\|_U \\ &\leq (1-\alpha^{-1}) && \text{Since } x \in U', \text{ so } \|x\|_U = 1 \\ &= \frac{\epsilon'}{2} < \epsilon' \leq \epsilon. \end{aligned}$$

All together, this states $\|x - \alpha^{-1}x\|_U < \epsilon$ for all $\epsilon > 0$. Since $\alpha^{-1}x \in M$, we have that $x \in \overline{M}$ and $x \in U$, as required. □

If E, F are normed spaces, a map $f : E \to F$ is *bounded* iff the following supremum exists

$$\|f\| = \sup\{\|f(x)\| \mid x \in E \text{ and } \|x\| \leq 1\}$$

Boundedness is equivalent to continuity for maps of normed spaces, and as indicated, the supremum above is a norm on continuous linear maps $E \to F$ [24, Proposition III.2.1].

Lemma 0.1.8. *Let E, F be real vector spaces and $U \subseteq E$, $V \subseteq F$ be absolutely convex absorbent sets such that $\|\text{-}\|_U$ and $\|\text{-}\|_V$ are norms. If $f : E \to F$ is a linear map such that $f(U) \subseteq V$, then $\|f\| \leq 1$. Consequently, if $f(U) \subseteq \alpha V$, where $\alpha \in \mathbb{R}_{>0}$, then $\|f\| \leq \alpha$, and so f is bounded.*

Proof. Let $x \in X$ and $\|x\|_U \leq 1$. By Lemma 0.1.6 this is equivalent to $x \in \alpha U$ for all $\alpha > 1$. We have by linearity of f that $f(\alpha U) \subseteq \alpha V$. Therefore $f(x) \in \alpha V$ for all $\alpha > 1$, so by applying Lemma 0.1.6 in reverse, we conclude that $\|f(x)\|_V \leq 1$. As this applies for all x such that $\|x\|_U \leq 1$, we can conclude that $\|f\| = \sup\{\|f(x)\|_V \mid x \in E.\|x\|_U \leq 1\} \leq 1$.

If $f(U) \subseteq \alpha V$, then $(\alpha^{-1}f)(U) \subseteq V$, so $\|\alpha^{-1}f\| \leq 1$, so $\|f\| \leq \alpha$. □

We require the following definitions and lemmas about Banach spaces. We say that a family $(x_i)_{i\in I}$ of elements of a Banach space E is *absolutely summable* if $\sum_{i\in I} \|x_i\|$ converges. (See [103, §1.4] or [118, Chapter III Exercise 23 (iii)].)

Lemma 0.1.9. *Let $(x_i)_{i\in I}$ be a family of nonnegative reals such that $\sum_{i\in I} x_i$ converges. Then the support of x_i is countable.*

Proof. The sum $\sum_{i\in I} x_i$ is defined to be

$$\lim_{S\in\mathcal{P}_{\text{fin}}(I)} \sum_{i\in S} x_i,$$

and we will give the value of the sum the short name λ. Since the sum converges, we have that for all $\epsilon > 0$ there exists $S_\epsilon \in \mathcal{P}_{\text{fin}}(I)$ such that $|\lambda - \sum_{i \in S_\epsilon} x_i| < \epsilon$. The set $S = \bigcup_{j=1}^{\infty} S_{2^{-j}}$ is a countable union of finite sets, hence countable. Suppose that there is some $i' \in I \setminus S$ such that $x_{i'} \neq 0$. Then, taking a j such that $x_i' > 2^{-j}$, we have that

$$\lambda - \sum_{i \in S_j} x_i < 2^{-j} < x_{i'},$$

and so $\sum_{i \in S_j} x_i + x_{i'} > \lambda$. But since λ is the sum of (x_i) over all $i \in I$ and $i' \notin S_j$, we have

$$\sum_{i \in S_i} x_i + x_{i'} \leq \lambda.$$

This contradicts our assumption that such an i' existed. Therefore the support of $(x_i)_{i \in I}$ is contained in S, and hence is countable. □

Corollary 0.1.10. *Let $(x_i)_{i \in I}$ be an absolutely summable family of elements of some normed space. The support of (x_i) is countable.*

Proof. Since $(x_i)_{i \in I}$ is absolutely summable, $(\|x_i\|)_{i \in I}$ is a summable sequence of nonnegative reals, and so has countable support by Lemma 0.1.9. Since $\|x_i\| = 0$ iff $x_i = 0$, $(x_i)_{i \in I}$ has the same, countable, support as $(\|x_i\|)_{i \in I}$. □

The above applies in particular to sums of real numbers, as $\mathbb{R}$ is a normed space. The corollary above can also be found as an exercise in [118, Chapter III Exercise 23 (c)].

Lemma 0.1.11. *In any Banach space E, every absolutely summable family $(x_i)_{i \in I}$ is summable,* i.e. $\sum_{i \in I} x_i$ *converges.*

Proof. Let $J = \mathcal{P}_{\text{fin}}(I)$, being a directed poset under inclusion, and $(S_j)_{j \in J}$ be the net defined as

$$S_j = \sum_{i \in j} x_i.$$

By definition, $\sum_{i \in I} x_i$ converges iff S_j converges. We show S_j converges by showing that it is Cauchy, *i.e.* that for all $\epsilon > 0$ there is a $N_\epsilon \in J$ such that for all $j, k \geq N_\epsilon$, $\|S_j - S_k\| < \epsilon$.

Let $\epsilon > 0$. We will consider the sum $\sum_{i \in I} \|x_i\|$. Define $(S'_j)_{j \in J}$ as

$$S'_j = \sum_{i \in j} \|x_i\|$$

Since $\sum_{i \in I} \|x_i\|$ converges, there is some $N_\epsilon \in J$ such that for all $j, k \geq N_\epsilon$, $|S'_j - S'_k| < \epsilon$. □

The previous lemma is found as an exercise in [118, Chapter III Exercise 23 (a)].

Lemma 0.1.12. *Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be absolutely summable families in a Banach space E, with the same index set I. Then $(x_i + y_i)_{i\in I}$ is absolutely summable and*

$$\sum_{i\in I}(x_i + y_i) = \sum_{i\in I} x_i + \sum_{i\in I} y_i.$$

Proof. By definition,

$$\sum_{i\in I} \|x_i + y_i\| = \lim_{S\in\mathcal{P}_{\text{fin}}(I)} \sum_{i\in S} \|x_i + y_i\|.$$

Since each term of the sum is non-negative, the net $\left(\sum_{i\in S} \|x_i + y_i\|\right)_{S\in\mathcal{P}_{\text{fin}}(I)}$ is monotone. If we observe that for all $S \in \mathcal{P}_{\text{fin}}(I)$

$$\sum_{i\in S} \|x_i + y_i\| \leq \sum_{i\in S} \|x_i\| + \|y_i\|$$

and that $\lim_{S\in\mathcal{P}_{\text{fin}}(I)} \sum_{i\in S} \|x_i\| + \|y_i\|$ exists by continuity of addition (for real numbers), we can use Lemma A.1.2 to conclude that $\sum_{i\in I} \|x_i + y_i\|$ converges. Therefore $\sum_{i\in I}(x_i + y_i)$ converges by Lemma 0.1.11, and so

$$\sum_{i\in I}(x_i + y_i) = \sum_{i\in I} x_i + \sum_{i\in I} y_i$$

by continuity of addition in E. □

A *locally convex space* is a Hausdorff topological vector space where the convex neighbourhoods of each point form a neighbourhood base of that point, equivalently that the absolutely convex neighbourhoods of 0 form a neighbourhood base for 0 [118, II.4]. It is equivalent to require that the absolutely convex neighbourhoods of 0 form a neighbourhood base for 0 [118, I.1.2]. Under the correspondence between absorbing absolutely convex sets and seminorms, locally convex spaces are also exactly those spaces whose topology can be defined by a separating family of seminorms, and this is sometimes used as a definition [24, Definition IV.1.2]. Products of locally convex spaces are given by the topological product, and neighbourhoods of zero of the form $U \times V$ for U a neighbourhood of zero in the left factor and V one in the right factor form a base [118, II.5.2 Products].

If E is a locally convex space, a subset $S \subseteq E$ is said to be *bounded* if S is absorbed by all neighbourhoods of zero. If $f : E \to F$ is a continuous linear map, then $f(S)$ is a bounded subset of F[118, I.5.4].

We include here some basic lemmas about bounded sets in locally convex spaces.

Lemma 0.1.13.

(i) A set $S \subseteq E$ is bounded in a topological vector space iff it is absorbed by all elements of a neighbourhood base for 0*.*

(ii) *If $X \subseteq E$, $Y \subseteq F$ are bounded subsets of locally convex spaces, then $X \times Y$ is a bounded subset of $E \times F$.*

(iii) *If $X \subseteq E$ is a bounded subset, $x \in E$, then $X + x$ is bounded.*

Proof.

(i) The only if direction is clear. We therefore show that if S is absorbed by all the elements of a neighbourhood base $\mathcal{N}$ for 0 in E, S is bounded. Let U be a neighbourhood of 0, and $N \in \mathcal{N}$ a basic neighbourhood such that $N \subseteq U$, which must exist by $\mathcal{N}$ being a neighbourhood base. By assumption, N absorbs S, so there is an $\alpha > 0$ such that $S \subseteq \alpha N$. Since $\alpha N \subseteq \alpha U$, we have that U absorbs S. Since this applies for an arbitrary neighbourhood of 0, S is bounded.

(ii) By part (i), it suffices to show that if $U \subseteq E$ and $V \subseteq F$ are 0-neighbourhoods in E and F respectively, that $U \times V$ absorbs $X \times Y$. We know there exist $\alpha, \beta > 0$ such that $X \subseteq \alpha U$ and $Y \subseteq \beta V$. Let $\gamma = \max\{\alpha, \beta\}$. Then we have $X \subseteq \gamma U$ and $Y \subseteq \gamma V$, so $X \times Y \subseteq \gamma(U \times V)$, as required.

(iii) By part (i), it suffices to show that for any absolutely convex neighbourhood U of 0, there is an $\alpha > 0$ such that $X + x \subseteq \alpha U$. Since X is bounded, there is a $\beta > 0$ such that $X \subseteq \beta U$, and as U is absorbent, there is a $\gamma > 0$ such that $x \subseteq \gamma U$. We can take $\alpha = \beta + \gamma$, *i.e.* $X + x \subseteq (\beta + \gamma)U$, because if $y \in X$, so $\beta^{-1}y, \gamma^{-1}x \in U$ so

$$\begin{aligned}\frac{y+x}{\beta+\gamma} &= \frac{y}{\beta+\gamma} + \frac{x}{\beta+\gamma} \\ &= \frac{\beta}{\beta+\gamma}\beta^{-1}y + \frac{\gamma}{\beta+\gamma}\gamma^{-1}x \in U\end{aligned}$$

by convexity of U. □

Lemma 0.1.14. *Every compact subset of a locally convex space is bounded.*

Proof. Let E be a locally convex space, with K a compact subset and U a 0-neighbourhood, and $V = \operatorname{int}(U)$, which is necessarily an open 0-nbhd. By Lemma 0.1.4, V is absorbent, so $\{\alpha V\}_{\alpha \in \mathbb{R}_{>0}}$ is an open cover of E, and therefore of K. Applying compactness, we take a finite subcover, if we take the largest $\beta \in \mathbb{R}_{>0}$ such that βV is in this subcover, it contains all the other sets in the subcover so $K \subseteq \beta V$. Therefore $K \subseteq U$, and so K is bounded. □

Lemma 0.1.15. *If $S \subseteq E$ is bounded, E being a locally convex space, then its absolutely convex hull* $\operatorname{absco}(S)$ *is also bounded.*

Proof. Using Lemma 0.1.13 and local convexity of E, we only need to show that $\operatorname{absco}(S)$ is absorbed by all absolutely convex neighbourhoods of 0. So let U be an absolutely convex 0-neighbourhood. We know that there is an $\alpha > 0$ such that $S \subseteq \alpha U$. We therefore have that $\operatorname{absco}(S) \subseteq \operatorname{absco}(\alpha U) = \alpha U$ as αU was absolutely convex to start off with. Therefore U absorbs $\operatorname{absco}(S)$ and so $\operatorname{absco}(S)$ is bounded. □

Lemma 0.1.16. *If $X \subseteq E$ is bounded, for E locally convex, then X is radially bounded.*

Proof. Suppose for a contradiction that X is bounded, but radially unbounded. By Lemma 0.1.15, $\text{absco}(X)$ is bounded, but also radially unbounded as it contains X. By Lemma 0.1.2 there exists a line through the origin in $\text{absco}(X)$, which we take to be generated by a nonzero element $x \in \text{absco}(X)$. The boundedness of $\text{absco}(X)$ implies that for each 0-neighbourhoods U, there is an $\alpha > 0$ such that $\text{absco}(X) \subseteq \alpha U$, and so $\beta x \in \text{absco}(X) \subseteq \alpha U$ for all $\beta \in \mathbb{R}$. This implies that $x \in \frac{\alpha}{\beta} U$ for all $\beta > 0$, so by taking $\beta = \alpha^{-1}$ we obtain $x \in U$ for any 0-neighbourhood U.

Since E is Hausdorff, there are open sets $U, V \subseteq E$ such that $0 \in U$, $x \in V$ and $U \cap V = \emptyset$. Therefore U does not contain x, contradicting the previous paragraph. We therefore have that X is radially bounded by contradiction. □

We also have to following lemma about products of locally convex spaces.

Lemma 0.1.17. *Let $E \times F$ be a product of locally convex spaces. We have a map $\kappa_1 : E \to E \times F$ defined as*

$$\kappa_1(x) = (x, 0).$$

This is a continuous linear section of π_1, and hence a linear homeomorphism onto its image $E \times 0$. The analogous statements are true for κ_2 and π_2.

Proof. We only give the proof for κ_1 and π_1 as the other side is analogous. We see that κ_1 is linear because addition and scalar multiplication in $E \times F$ are pointwise. If $U \times V$ is a basic neighbourhood of 0 in $E \times F$, then $\kappa_1^{-1}(U \times V) = U$, so κ_1 is continuous. We can see that it is a section of π_1 because

$$\pi_1 \circ \kappa_1(x) = \pi_1(x, 0) = x$$

for all $x \in E$. This implies that it is a homeomorphism onto its image in $E \times F$. □

We now deal with some notions relating to completeness in locally convex spaces. In any topological vector space E, we can define a uniformity[2] on E by taking a base of entourages to be the family of sets of the form

$$N_V = \{(x, y) \in E \times E \mid x - y \in V\}$$

where V runs through some base of 0-neighbourhoods [118, §I.1.4]. The topology defined by this uniformity is the topology E started with, and whenever we apply a notion relating to uniform spaces to topological vector spaces, we mean to use this uniformity. Any continuous linear map is uniformly continuous. A topological vector space is said to be complete iff it is complete in that uniformity.

[2] See [19, §II.1.1] for the basic theory of uniform spaces.

Lemma 0.1.18. *For any locally convex space E, a an element of E, the map $- + a : E \to E$ is an affine uniform isomorphism.*

Proof. We first prove that $- + a$ is affine and uniformly continuous. Consider a convex combination $\alpha x + (1-\alpha)y$ in E in the following:

$$\begin{aligned}(\alpha x + (1-\alpha)y) + a &= \alpha x + (1-\alpha)y + \alpha a + (1-\alpha)a \\ &= \alpha(x+a) + (1-\alpha)(y+a).\end{aligned}$$

To show it is uniformly continuous, let N_V be a basic entourage coming from a 0-neighbourhood V. We will show that $N_V \subseteq ((- + a) \times (- + a))^{-1}(N_V)$, equivalently $((- + a) \times (- + a))(N_V) \subseteq N_V$ as follows. Let $(x, y) \in N_V$, *i.e.* $x - y \in V$. Then

$$((- + a) \times (- + a))(x, y) = (x + a) - (y + a) = x - y \in V$$

so $(x + a, y + a) \in N_V$.

We then observe that $-+(-a)$ is of the same form, hence affine and uniformly continuous, and the inverse to $- + a$, so $- + a$ is a uniform isomorphism. □

A sequence $(x_i)_{i\in\mathbb{N}}$ in X is a *Cauchy sequence* if for each entourage $U \subseteq X \times X$, there is an $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have $(x_i, x_j) \in U$. So in a topological vector space, a sequence is Cauchy if for each 0-neighbourhood (equivalently, for each basic 0-neighbourhood for some 0-neighbourhood base) U, there is an $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have $x_i - x_j \in U$. If we consider a normed space as a locally convex space, with its 0-neighbourhood base of open balls of radius ϵ, we see that this coincides with the usual notion of Cauchy sequence in normed spaces. A subset S of a locally convex space E is *sequentially complete* if every Cauchy sequence with values in S converges to a point in S.

Lemma 0.1.19. *Let $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of real numbers, $\alpha_i \geq 0$ for all i, such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in a locally convex space E that is bounded (as a subset of E). Then the sequence*

$$\left(\sum_{i=1}^{n} \alpha_i x_i\right)_{n\in\mathbb{N}}$$

is Cauchy.

Proof. Let $U \subseteq E$ be an absolutely convex neighbourhood of 0. Since $(x_i)_{i\in\mathbb{N}}$ is bounded, there is a $\beta > 0$ such that $x_i \in \beta U$, or equivalently $\frac{1}{\beta} x_i \in U$ for all $i \in \mathbb{N}$. Since $\left(\sum_{i=1}^{n} \alpha_i\right)_{n\in\mathbb{N}}$ converges, as the sum of that series is 1, it is a Cauchy sequence, so there is an $N \in \mathbb{N}$ such that for all $m, n \geq N$ (without loss of generality taking $m \geq n$) we have

$$\left|\sum_{i=n+1}^{m} \alpha_i\right| < \frac{1}{\beta},$$

and since each term of the sum is nonnegative

$$0 \leq \sum_{i=n+1}^{m} \alpha_i < \frac{1}{\beta},$$

and in fact

$$0 \leq \sum_{i=n+1}^{m} \beta\alpha_i < 1.$$

We can now see that for all $n, m \geq N$, without loss of generality taking $m \geq n$, we have

$$\sum_{i=1}^{m} \alpha_i x_i - \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=n+1}^{m} \alpha_i x_i = \sum_{i=n}^{m} (\beta\alpha_i) \left(\frac{1}{\beta} x_i\right).$$

As this is an absolutely convex combination of elements of U, we have shown that $\sum_{i=1}^{m} \alpha_i x_i \in U$, as is required to show that the sequence is Cauchy. □

0.2 Ordered Vector Spaces

A *wedge* in a (real) vector space E is a subset E_+ such that:

(i) If $x, y \in E_+$, $x + y \in E_+$.

(ii) If $\alpha \in \mathbb{R}$, $\alpha \geq 0$, and $x \in E_+$, then $\alpha x \in E_+$.

The wedge defines a pre-order on E by defining

$$x \leq y \Leftrightarrow y - x \in E_+. \tag{1}$$

In fact, this defines a map from vector spaces with a wedge to vector spaces that are also pre-ordered sets where the pre-order is translation invariant. Taking the wedge of elements greater than or equal to zero defines the inverse map, so these two structures are equivalent.

We say that a wedge is a *cone* if $E_+ \cap -E_+ = \{0\}$. Some authors call this a *proper cone* and use cone, or even *convex cone* to mean a wedge, reserving cone for an even more general notion. We will stick to the previous terminology. For a cone, (1) defines a partial order. We will now use *partially ordered vector space* to refer to a pair (E, E_+). In fact, we often omit the word partially and refer to these simply as *ordered vector spaces*. A linear map $f : E \to F$ between partially ordered vector spaces (E, E_+) and (F, F_+) is *positive* if $f(E_+) \subseteq F_+$. For linear maps, this is equivalent to being monotone in the order (1). Partially ordered vector spaces and linear maps form a category.

We say a poset P is *directed* if it is nonempty and each pair has an upper bound, *i.e.* if for each pair $x, y \in P$ there exists $z \in P$ such that $x \leq z$ and $y \leq z$. We say a cone $E_+ \subseteq E$ is *generating* if E is the (real) span of E_+, equivalently $E_+ - E_+ = E$. This is equivalent to the statement that each $x \in E$ can be (nonuniquely, in general) expressed as $x_+ - x_-$ with $x_+, x_- \in E_+$. Many authors say instead that (E, E_+) is directed, for the following reason.

Proposition 0.2.1. *A partially ordered vector space (E, E_+) is directed iff E_+ is generating.*

Proof.

- Directed implies generating:

  Let $x \in E$. Since (E, E_+) is directed, there exists an element, which we shall call x_+, such that $x \leq x_+$ and $0 \leq x_+$. Applying (1), we see that $x_+ \in E_+$ and $x_+ - x \in E_+$. If we define $x_- = x_+ - x$, we see that $x = x_+ - x_-$ and $x_+, x_- \in E_+$, as required.

- Generating implies directed:

  Let $x, y \in E$. We can choose decompositions of them into positive elements as $x_+ - x_- = x$ and $y_+ - y_- = y$. These equations can also be written as

  $$x_+ - x = x_- \qquad\qquad y_+ - y = y_-,$$

  which by (1) implies $x \leq x_+$ and $y \leq y_+$. Using the fact that $x_+, y_+ \in E_+$ and the translation invariance of the order, we also have

  $$x_+ \leq x_+ + y_+ \qquad\qquad y_+ \leq x_+ + y_+,$$

  so we can apply transitivity of $\leq$ to deduce

  $$x \leq x_+ + y_+ \qquad\qquad y \leq x_+ + y_+.$$

  We can therefore see that $x_+ + y_+$ is an upper bound for $\{x, y\}$. Since x and y are arbitrary, and E can never be empty, E is directed. □

From now on we will use the common terminology and refer to (E, E_+) as directed if E_+ is generating. We remark at this point that $(\mathbb{R}, [0, \infty))$ is a directed ordered vector space and the order is the usual one.

We can extend the notation for closed intervals, as used on $\mathbb{R}$, to any ordered vector space. If (E, E_+) is an ordered vector space and $a, b \in E$ is any pair of elements, we define

$$\begin{aligned} [a, b] &= \{x \in E \mid x - a \in E_+ \text{ and } b - x \in E_+\} \\ [a, \infty) &= \{x \in E \mid x - a \in E_+\} = E_+ + a \\ (-\infty, b] &= \{x \in E \mid b - x \in E_+\} = b - E_+. \end{aligned}$$

It is clear from these definitions that $[a, b] = [a, \infty) \cap (-\infty, b] = a + E_+ \cap b - E_+$.

Lemma 0.2.2. *Let (E, E_+) be an ordered vector space, a, b elements of E.*

(i) If $\alpha \in \mathbb{R}_{>0}$. Then

$$\alpha[a, b] = [\alpha a, \alpha b]$$

(ii) If $c \in E$, then

$$c + [a, b] = [a + c, b + c]$$

Proof.

(i) We reason as follows:

$$\begin{aligned} x \in \alpha[a,b] &\Leftrightarrow \alpha^{-1}x \in [a,b] \\ &\Leftrightarrow \alpha^{-1}x - a \in E_+ \text{ and } b - \alpha^{-1}x \in E_+ \\ &\Leftrightarrow x - \alpha a \in E_+ \text{ and } \alpha b - x \in E_+ && E_+ \text{ a cone} \\ &\Leftrightarrow x \in [-\alpha a, \alpha b]. \end{aligned}$$

(ii) In this case:

$$\begin{aligned} x \in c + [a,b] &\Leftrightarrow x - c \in [a,b] \\ &\Leftrightarrow a \leq x - c \leq b \\ &\Leftrightarrow x - c - a \in E_+ \text{ and } b - x + c \in E_+ \\ &\Leftrightarrow x - (a+c) \in E_+ \text{ and } (b+c) - x \in E_+ \\ &\Leftrightarrow x \in [a+c, b+c] \end{aligned}$$

so the two sets are the same. □

0.3 Dualities, Polars and Bipolars

A *duality* is a triple $(E, F, \langle\text{-},\text{-}\rangle)$ where E, F are real vector spaces, and $\langle\text{-},\text{-}\rangle : E \times F \to \mathbb{R}$ is a bilinear map such that

$$\begin{aligned} \forall y \in F. \langle x, y\rangle = 0 \text{ implies } x = 0 \\ \forall x \in E. \langle x, y\rangle = 0 \text{ implies } y = 0. \end{aligned}$$

Some authors use *separated duality* to describe this, leaving the term duality to refer to a pair of vector spaces E, F with a bilinear map $E \times F \to \mathbb{R}$. The basic theory is described in [118, §IV.1] and [18, §II.6.1]. By the symmetry in the definition, if $(E, F, \langle\text{-},\text{-}\rangle)$ is a duality, $(F, E, \langle\text{-},\text{-}\rangle \circ \sigma_{E,F})$ is a duality, where $\sigma_{E,F}(x, y) = (y, x)$. We call this the *transpose* of a duality.

For any locally convex topological vector space E, we denote by E^* the vector space of k-valued continuous linear maps, the *dual space*, where $k \in \{\mathbb{R}, \mathbb{C}\}$ is the base field of E. This is also used by some authors for the "algebraic dual", of all linear maps, including discontinuous ones, who use E' for the continuous dual. However, E' is used to refer to the commutant in the theory of operator algebras, so we do not use it.

Proposition 0.3.1. *If E is a locally convex space, and we define*

$$\begin{aligned} \langle\text{-},\text{-}\rangle &: E \times E^* \to \mathbb{R} \\ \langle x, \phi\rangle &= \phi(x) \end{aligned}$$

then $(E, E^*, \langle\text{-},\text{-}\rangle)$ is a duality.

Proof. See [118, §IV.1 Example 2] or [18, p. II.42]. □

We can define a locally convex topology $\sigma(E, F)$ on E with the following subbase of zero-neighbourhoods

$$N_{y,\epsilon} = \{x \in X \mid |\langle x, y\rangle| < \epsilon\}. \tag{2}$$

Where $y \in F$ and $\epsilon \in \mathbb{R}_{>0}$. In fact, $(N_y) = (N_{y,1})_{y \in F}$ defines the same topology. This topology is the coarsest topology such that each $\langle \text{-}, y\rangle : E \to \mathbb{R}$ is continuous. By transposing the duality, we can also define the topology $\sigma(F, E)$ on F. In the special case of the pairing from Proposition 0.3.1, $\sigma(E, E^*)$ is called the *weak topology* and $\sigma(E^*, E)$ the *weak-* topology*. If $(E, F, \langle \text{-}, \text{-}\rangle)$ is a pairing, and $Y \subseteq F$ is a set whose span is F, then we can define $\sigma(E, Y)$ to be the topology with sets of the form $N_{y,\epsilon}$ with $y \in Y$ as a base. This topology agrees with $\sigma(E, F)$.

We will need the following fundamental results about linear maps that are continuous in weak topologies.

Proposition 0.3.2. *Let $(E, F, \langle \text{-}, \text{-}\rangle)$ be a duality. The map $x \mapsto \langle x, \text{-}\rangle$ defines a linear isomorphism from E to $(F, \sigma(F, E))^*$,* i.e. *from E to linear maps from F to $\mathbb{R}$ that are continuous in the $\sigma(F, E)$-topology. By symmetry, the map $y \mapsto \langle \text{-}, y\rangle$ defines a linear isomorphism from F to $(E, \sigma(E, F))^*$.*

Proof. See [118, IV.1.2]. □

Proposition 0.3.3. *Let $(E_1, F_1, \langle \text{-}, \text{-}\rangle_1)$ and $(E_2, F_2, \langle \text{-}, \text{-}\rangle_2)$ be dualities. Let $f : E_1 \to E_2$ be a linear map. The following statements are equivalent:*

(i) f is continuous from $\sigma(E_1, F_1)$ to $\sigma(E_2, F_2)$.

(ii) There exists a linear map $g : F_2 \to F_1$ such that for all $x \in E_1$ and $y \in F_2$

$$\langle f(x), y\rangle_2 = \langle x, g(y)\rangle_1 \tag{3}$$

Any g satisfying (3) is necessarily continuous from $\sigma(F_2, E_2)$ to $\sigma(F_1, E_1)$.

Proof. See [118, IV.2.1]. □

Proposition 0.3.4. *Let E be a locally convex space, and C a convex subset. Then C is closed iff it is $\sigma(E, E^*)$-closed, and the closure of C is the $\sigma(E, E^*)$-closure.*

Proof. See [118, II.9.2 Corollary 2]. □

0.3.1 Polars

Given a duality $(E, F, \langle \text{-}, \text{-} \rangle)$, and a subset $S \subseteq E$, we define the *polar* [118, §IV.1.3] of S, $S^{\mathrm{o}} \subseteq F$, to be

$$S^{\mathrm{o}} = \{y \in F \mid \forall x \in S.\langle x, y \rangle \leq 1\}$$

If $T \subseteq F$, we define T^{o} to be the polar of T in the transposed duality.

In [18, §II.6.3] the polar is defined to be

$$\{y \in F \mid \forall x \in S.\langle x, y \rangle \geq -1\}.$$

It is clear that this is $-S^{\mathrm{o}}$, and so any results about polars from [18] can be translated to match the usual definitions.

We define the *absolute polar* as

$$S^{|\mathrm{o}|} = \{y \in F \mid \forall x \in S.|\langle x, y \rangle| \leq 1\}.$$

Lemma 0.3.5. *Let $(E, F, \langle \text{-}, \text{-} \rangle)$ be a pairing and $S \subseteq E$. Then S^{o} is a convex subset of F containing 0 that is $\sigma(F, E)$-closed, and $S^{|\mathrm{o}|}$ is absolutely convex and $\sigma(F, E)$-closed.*

Proof. For the fact that S^{o} is convex, contains 0, and is $\sigma(F, E)$-closed, see [118, IV.1.4].

To show $S^{|\mathrm{o}|}$ is absolutely convex, let $\sum_{i \in I} \alpha_i y_i$ be a finite absolutely convex combination of elements of $S^{|\mathrm{o}|}$. Then for all $x \in S$, we have

$$\left| \left\langle x, \sum_{i \in I} \alpha_i y_i \right\rangle \right| = \left| \sum_{i \in I} \alpha_i \langle x, y_i \rangle \right| \leq \sum_{i \in I} |\alpha_i| |\langle x, y_i \rangle| \leq \sum_{i \in I} |\alpha_i| \leq 1.$$

This shows $\sum_{i \in I} \alpha_i y_i \in S^{|\mathrm{o}|}$. Now, since $S^{|\mathrm{o}|} = S^{\mathrm{o}} \cap -S^{\mathrm{o}}$, it is $\sigma(F, E)$-closed as well. □

Lemma 0.3.6. *Let $(E, F, \langle \text{-}, \text{-} \rangle)$ be a pairing. If $S \subseteq E$ is absolutely convex, then $S^{\mathrm{o}} = S^{|\mathrm{o}|}$.*

Proof. We see from the definition that $S^{|\mathrm{o}|} \subseteq S^{\mathrm{o}}$. To show the opposite inclusion, let $y \in S^{\mathrm{o}}$. We know that for all $x \in S$, $\langle x, y \rangle \leq 1$. Since $-x \in S$, by absolute convexity, we have $\langle -x, y \rangle \leq 1$, so $\langle x, y \rangle \geq -1$, by bilinearity. This shows that $|\langle x, y \rangle| \leq 1$, for all $x \in S$, and hence $y \in S^{|\mathrm{o}|}$. □

Recall that the polar wedge of a wedge $C \subseteq E$ is

$$C^* = \{y \in F \mid \langle x, y \rangle \geq 0\}$$

Lemma 0.3.7. *Let $(E, F, \langle \text{-}, \text{-} \rangle)$ be a pairing, and $C \subseteq E$ a wedge. Then $C^* = -C^{\mathrm{o}}$.*

Proof. We have that

$$\begin{aligned}-C^{\mathrm{o}} &= -\{y \in F \mid \forall x \in C.\langle x, y\rangle \leq 1\} = \{y \in F \mid \forall x \in C.\langle x, -y\rangle \leq 1\} \\ &= \{y \in F \mid \forall x \in C.\langle x, y\rangle \geq -1\}.\end{aligned}$$

We can see, therefore, that $-C^{\mathrm{o}} \subseteq C^*$. Suppose for a contradiction that there is a $y \in -C^{\mathrm{o}} \setminus C^*$. Then there is some $x \in C$ such that $-1 \leq \langle x, y\rangle < 0$. Take $\alpha = \langle x, y\rangle$. Therefore $-\frac{2}{\alpha} > 0$, so $-\frac{2}{\alpha}x \in C$ because it is a wedge. We can therefore see that

$$\left\langle -\frac{2}{\alpha}x, y\right\rangle = -\frac{2}{\alpha}\langle x, y\rangle = -2 \not\geq -1,$$

which is a contradiction. So $-C^{\mathrm{o}} \setminus C^* = \emptyset$, and therefore $-C^{\mathrm{o}} = C^*$. □

Lemma 0.3.8. *If $E = E_+ - E_+$ (equivalently, if (E, E_+) is directed), then the dual wedge F_+ is a cone, the* dual cone*.*

Proof. Suppose $y \in F_+$ and $-y \in F_+$. Then for all $x \in E_+$, we have $\langle x, y\rangle \geq 0$ and $\langle x, -y\rangle \geq 0$. By linearity, we deduce $\langle x, y\rangle \leq 0$ and therefore $\langle x, y\rangle = 0$ for all $x \in E_+$. If $x \in E$ is expressed as $x_+ - x_-$ with $x_+, x_- \in E_+$, we see that

$$\langle x, y\rangle = \langle x_+ - x_-, y\rangle = \langle x_+, y\rangle - \langle x_-, y\rangle = 0,$$

and therefore $y = 0$. □

The main theorem in the subject of polars is the following. Given a set $S \subseteq E$, we may not only take the polar S^{o}, but also the polar of the polar, $S^{\mathrm{oo}} \subseteq E$, the *bipolar*.

Theorem 0.3.9 (Bipolar Theorem)**.** *Let $(E, F, \langle \text{-}, \text{-}\rangle)$ be a duality, $S \subseteq E$ a subset. Then S^{oo} is the closed convex hull of $S \cup \{0\}$ in the $\sigma(E, F)$ topology.*

Proof. See [118, IV.1.5] or [18, II.6.3 Theorem 1]. □

Corollary 0.3.10. *Let $(E, F, \langle \text{-}, \text{-}\rangle)$ be a duality, and $S \subseteq E$ an absolutely convex set. Then S^{oo} is the $\sigma(E, F)$ closure of S, or equivalently the closed absolutely convex hull of S.*

Proof. By absolute convexity, we have $S \cup \{0\} = S$ and $\mathrm{co}(S \cup \{0\}) = S$. By [24, IV.1.13 Corollary], we have that the closed convex hull of S i the closure of the convex hull of S, so by the bipolar theorem $S^{\mathrm{oo}} = \mathrm{cl}\,(S)$, in the $\sigma(E, F)$ topology.

The closure of an absolutely convex set is convex, and if $x_i \to x$, then $-x_i \to -x$, so $\mathrm{cl}\,(S)$ is balanced and convex, therefore absolutely convex (Lemma A.3.1). Therefore $\mathrm{cl}\,(S)$ is the closed absolutely convex hull of S. □

Lemma 0.3.11. *Let $(E, F, \langle \text{-}, \text{-}\rangle)$ be a pairing.*

(i) Let $(S_i)_{i\in I}$ be a family of subsets of E. Then $\bigcap_{i\in I} {S_i}^{\mathrm{o}} = \left(\bigcup_{i\in I} S_i\right)^{\mathrm{o}}$, and $\bigcap_{i\in I} {S_i}^{|\mathrm{o}|} = \left(\bigcup_{i\in I} S_i\right)^{|\mathrm{o}|}$.

(ii) *Let* $S \subseteq E$ *and* $\alpha \in \mathbb{R} \setminus \{0\}$. *Then* $\alpha(S)^{\mathrm{o}} = (\alpha^{-1}S)^{\mathrm{o}}$ *and* $\alpha(S)^{|\mathrm{o}|} = (\alpha^{-1}S)^{|\mathrm{o}|}$.

(iii) *Let* $S, T \subseteq E$. *Then* $S \subseteq T$ *implies* $T^{\mathrm{o}} \subseteq S^{\mathrm{o}}$ *and* $T^{|\mathrm{o}|} \subseteq S^{|\mathrm{o}|}$.

(iv) *Let* $S \subseteq E$. *Then* $\mathrm{absco}S^{\mathrm{o}} = \mathrm{absco}(S)^{|\mathrm{o}|} = S^{|\mathrm{o}|}$.

Proof. In each of the first three statements, we only give the argument for the polar, as the argument for the absolute polar is similar.

(i)

$$\begin{aligned} y \in \bigcap_{i \in I} S_i{}^{\mathrm{o}} \Leftrightarrow \forall i \in I. \forall x \in S_i. \langle x, y \rangle \leq 1 \Leftrightarrow \forall x \in \bigcup_{i \in I} S_i. \langle x, y \rangle \leq 1 \\ \Leftrightarrow y \in \left(\bigcup_{i \in I} S_i \right)^{\mathrm{o}}. \end{aligned}$$

(ii)

$$\begin{aligned} y \in \alpha(S^{\mathrm{o}}) \Leftrightarrow \alpha^{-1} y \in S^{\mathrm{o}} \Leftrightarrow \forall x \in S. \langle x, \alpha^{-1} y \rangle \leq 1 \Leftrightarrow \forall x \in S. \langle \alpha^{-1} x, y \rangle \leq 1 \\ \Leftrightarrow \forall x \in \alpha^{-1} S. \langle x, y \rangle \leq 1 \Leftrightarrow y \in (\alpha^{-1} S)^{\mathrm{o}} \end{aligned}$$

(iii) Let $S \subseteq T \subseteq E$. Then

$$y \in T^{\mathrm{o}} \Leftrightarrow \forall x \in T. \langle x, y \rangle \leq 1 \Rightarrow \forall x \in S. \langle x, y \rangle \leq 1 \Leftrightarrow y \in S^{\mathrm{o}}.$$

(iv) By Lemma 0.3.6, $\mathrm{absco}S^{\mathrm{o}} = \mathrm{absco}S^{|\mathrm{o}|}$. We have $S \subseteq \mathrm{absco}(S)$, so by the previous part $\mathrm{absco}(S)^{|\mathrm{o}|} \subseteq S^{|\mathrm{o}|}$. To show the opposite inclusion, suppose that $y \in S^{|\mathrm{o}|}$, *i.e.* that for all $x \in S$, $|\langle x, y \rangle| \leq 1$. Let $\sum_{i \in I} \alpha_i x_i$ be a finite absolutely convex combination of elements of S. Then

$$\begin{aligned} \left| \left\langle \sum_{i \in I} \alpha_i x_i, y \right\rangle \right| &= \left| \sum_{i \in I} \alpha_i \langle x_i, y \rangle \right| \\ &\leq \sum_{i \in I} |\alpha_i \langle x_i, y \rangle| \\ &= \sum_{i \in I} |\alpha_i| |\langle x_i, y \rangle| \\ &\leq \sum_{i \in I} |\alpha_i| && x_i \in S,\ y \in S^{|\mathrm{o}|} \\ &\leq 1 && \text{an absolutely convex combination} \end{aligned}$$

□

Corollary 0.3.12. *If* $(E, F, \langle \text{-}, \text{-} \rangle)$ *is a duality, and* $S \subseteq E$, *then* $S^{|\mathrm{o}||\mathrm{o}|}$ *is the closed absolutely convex hull of* S *in the* $\sigma(E, F)$ *topology.*

Proof. By Lemma 0.3.11, we have $S^{|\mathrm{o}||\mathrm{o}|} = \mathrm{absco}(S)^{|\mathrm{o}||\mathrm{o}|}$, which in turn is equal to $\mathrm{absco}(S)^{\mathrm{oo}}$ by Lemma 0.3.6. By Corollary 0.3.10, this is the closed absolutely convex hull of S in the $\sigma(E, F)$ topology. □

Lemma 0.3.13. *Let $(E, F, \langle\text{-}, \text{-}\rangle)$ be a duality, and $F' \subseteq F$ a subspace of F such that F separates the points of E, and therefore $(E, F', \langle\text{-}, \text{-}\rangle)$ is a pairing. Let $S \subseteq E$. We use S^{o}_F to mean the polar of S in F and $S^{\mathrm{o}}_{F'}$ to mean the polar of S in F'. Then*

$$S^{\mathrm{o}}_F \cap F' = S^{\mathrm{o}}_{F'}$$

Proof. If $y \in S^{\mathrm{o}}_{F'}$, then $y \in F'$ and $\forall x \in S.\langle x, y\rangle \leq 1$, so $y \in S^{\mathrm{o}}_F$. Therefore $y \in S^{\mathrm{o}}_F \cap F'$.

For the other direction, let $y \in S^{\mathrm{o}}_F \cap F'$. Then $y \in F'$ and $\forall x \in S.\langle x, y\rangle \leq 1$, so $y \in S^{\mathrm{o}}_{F'}$. □

The following lemma relates polars and adjoints.

Lemma 0.3.14. *Let $(E_1, F_1, \langle\text{-}, \text{-}\rangle_1)$ and $(E_2, F_2, \langle\text{-}, \text{-}\rangle_2)$ be dualities, $f : E_1 \to E_2$ a linear map with adjoint $g : F_2 \to F_1$. If $S \subseteq E_1$, then $f(S)^{\mathrm{o}} = g^{-1}(S^{\mathrm{o}})$. Equivalently, if $T \subseteq F_2$, then $g(T)^{\mathrm{o}} = f^{-1}(T^{\mathrm{o}})$.*

Proof. The second statement follows from the first by transposing the duality, so we prove the first.

$$\begin{aligned} f(S)^{\mathrm{o}} &= \{y \in F_2 \mid \forall x \in f(S).\langle x, y\rangle_2 \leq 1\} \\ &= \{y \in F_2 \mid \forall x \in S.\langle f(x), y\rangle_2 \leq 1\} \\ &= \{y \in F_2 \mid \forall x \in S.\langle x, g(y)\rangle_1 \leq 1\} \\ &= \{y \in F_2 \mid g(y) \in S^{\mathrm{o}}\} \\ &= g^{-1}(S^{\mathrm{o}}). \end{aligned}$$

□

We can prove the following useful fact about closed wedges in locally convex spaces. It can be proven directly from the Hahn-Banach separation theorem as well.

Lemma 0.3.15. *Let E be a locally convex space and $E_+ \subseteq E$ a closed wedge. Then $\phi(x) \geq 0$ for all $\phi \in E^*_+$ (the polar wedge), iff $x \in E_+$.*

Proof. If $x \in E_+$, by definition we have $\phi(x) \geq 0$ for all $\phi \in E^*_+$. We therefore only need to show the other direction.

The set $\{x \in E \mid \forall\phi \in E^*_+.\phi(x) \geq 0\}$ is equal to the dual cone of E^*_+ with respect to the transpose of the usual pairing between E and E^*. Applying Lemma 0.3.7 twice, we have

$$\begin{aligned} \{x \in E \mid \forall\phi \in E^*_+.\phi(x) \geq 0\} &= -(-{E_+}^{\mathrm{o}})^{\mathrm{o}} \\ &= {E_+}^{\mathrm{oo}} \qquad \text{Lemma 0.3.11 (ii)} \end{aligned}$$

The bipolar is the $\sigma(E, E^*)$-closure of E_+ by the bipolar theorem and the fact that a wedge is already convex and contains 0. We then use the fact that if a convex set is closed in a locally convex space it is also weakly closed by Proposition 0.3.4. ☐

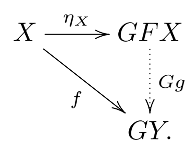
0.4 Category Theory

We recall here some basic theorems of category theory. Some basic references are [83, 17]. We use Eilenberg's notation for hom sets in a category, *i.e.* if $\mathcal{C}$ is a (locally small) category X, Y are objects in $\mathcal{C}$, then $\mathcal{C}(X, Y)$ is the set of arrows $X \to Y$ in $\mathcal{C}$.

The basic definition of an adjunction is a pair of functors $F : \mathcal{D} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $\phi : \mathcal{D}(F(X), Y) \Rightarrow \mathcal{C}(X, G(Y))$. An adjunction can be defined equivalently in the following ways:

Theorem 0.4.1. *Each adjunction (F, G, ϕ), $F : \mathcal{D} \to \mathcal{C}$ is determined by any one of the following:*

(i) *Functors F, G, a natural transformation $\eta : \mathrm{Id} \Rightarrow GF$ such that each η_X is a universal arrow from X to G,* i.e. *for each $f : X \to GY$ there exists a unique $g : FX \to Y$ such that the following diagram commutes*



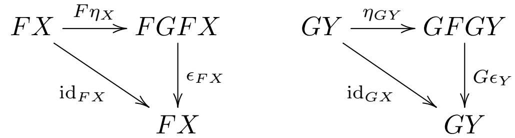
In this case, $\phi(f)$ is defined to be $G(f) \circ \eta_X$.

(ii) *The functor G and for each $X \in \mathcal{D}$ an object $F_0X \in \mathcal{C}$ and a universal arrow $\eta_X : X \to GF_0X$ from X to G. Then F is defined on objects as F_0X and on maps $h : X \to X'$ to be the unique Fh such that $GFh \circ \eta_X = \eta_{X'} \circ h$.*

(iii) *Functors F, G and a natural transformation $\epsilon : FG \Rightarrow \mathrm{Id}$ such that ϵ_Y is universal from F to Y.*

(iv) *The functor F and for each $Y \in \mathcal{C}$ an object $G_0Y \in \mathcal{D}$ and an arrow $\epsilon_Y : FG_0Y \to Y$ that is universal from F to Y.*

(v) *Functors F, G and natural transformations $\eta : \mathrm{Id} \Rightarrow GF$, $\epsilon : FG \Rightarrow \mathrm{Id}$ such that the following diagrams commute*



for each $X \in \mathcal{D}$ and $Y \in \mathcal{C}$.

Proof. See [83, IV.1 Theorem 2], where the statement of the theorem comes from. □

The following fact about adjoints justifies referring to "the" left adjoint or "the" right adjoint of a functor, at least up to isomorphism.

Proposition 0.4.2. *If $F, F' : \mathcal{D} \to \mathcal{C}$ are both left adjoints to a functor $G : \mathcal{C} \to \mathcal{D}$, then $F \cong F'$. Similarly, if G, G' are both right adjoints to a functor F, then $G \cong G'$.*

Proof. See [83, §IV.1 Corollary 1]. This isomorphism can be defined in terms of the units $\eta : \mathrm{Id} \Rightarrow GF, \eta' : \mathrm{Id} \Rightarrow GF'$ and counits $\epsilon : FG \Rightarrow \mathrm{Id}, \epsilon' : F'G \Rightarrow \mathrm{Id}$ as $\epsilon F' \circ F\eta' : F \to F'$ and $\epsilon' F \circ F'\eta : F' \to F$. These can be seen to be mutually inverse by using naturality and the triangle identities from Theorem 0.4.1 (v). □

A functor $F : \mathcal{D} \to \mathcal{C}$ is an *equivalence* if there exist $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha : FG \Rightarrow \mathrm{Id}_{\mathcal{C}}, \beta : \mathrm{Id}_{\mathcal{D}} \Rightarrow GF$. We call the triple (G, α, β) an *inverse* for F, and when no confusion can result, we refer to just G as an inverse. Be warned that there is no uniqueness to α and β. An adjunction (F, G, η, ϵ), described in terms of (v) of the above theorem, is called an *adjoint equivalence* if η and ϵ are isomorphisms. Recall that a functor F is *essentially surjective on objects* if for each $Y \in \mathcal{C}$, there exists an $X \in \mathcal{D}$ and an isomorphism $F(X) \cong Y$.

Theorem 0.4.3. *The following are equivalent for a functor $F : \mathcal{D} \to \mathcal{C}$:*

(i) F is an equivalence of categories.

(ii) F is part of an adjoint equivalence (F, G, η, ϵ). (and equally $(G, F, \epsilon^{-1}, \eta^{-1})$ is an adjoint equivalence).

(iii) F is full, faithful and essentially surjective on objects.

Proof. See [83, IV.4 Theorem 1]. □

For F an equivalence, we call any G in an adjoint equivalence an *adjoint inverse* to F.

Lemma 0.4.4. *Let (F, G, η, ϵ) be an adjoint equivalence, and let $(F', G, \eta', \epsilon')$ be an adjunction (having the same G). Then η' and ϵ' are isomorphisms, so $(F', G, \eta', \epsilon')$ is also an adjoint equivalence (with $F' \cong F$).*

Proof. Using Proposition 0.4.2, we have that $\epsilon F' \circ F\eta' : F \Rightarrow F'$ and $\epsilon' F \circ F'\eta : F' \Rightarrow F$ form an isomorphism $F \cong F'$. We first observe that

$$\begin{aligned} G(\epsilon F' \circ F\eta') \circ \eta &= G\epsilon F' \circ GF\eta' \circ \eta \\ &= G\epsilon F' \circ \eta GF' \circ \eta' && \text{naturality of } \eta \\ &= (G\epsilon \circ \eta G)F' \circ \eta' \\ &= \eta' && \text{adjunction triangle.} \end{aligned}$$

As η is an isomorphism, η' has been shown to be a composite of two isomorphisms, and therefore an isomorphism.

Similarly the proof for ϵ' goes

$$\begin{aligned} \epsilon \circ (\epsilon' F \circ F'\eta)G &= \epsilon \circ \epsilon' FG \circ F'\eta G \\ &= \epsilon' \circ F'G\epsilon \circ F'\eta G \\ &= \epsilon' \circ F'(G\epsilon \circ \eta G) \\ &= \epsilon', \end{aligned}$$

using the naturality of ϵ' and an adjunction triangle.

It follows, by definition, that $(F', G, \eta', \epsilon')$ is an adjoint equivalence. □

Lemma 0.4.5. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, $\alpha : F \Rightarrow G$ a natural isomorphism. Then for each parallel pair of maps $f, g : X \to Y$ in $\mathcal{C}$, we have*

$$F(f) = F(g) \Leftrightarrow G(f) = G(g).$$

Proof. By the naturality of α, we have $F(f) = \alpha_Y^{-1} \circ G(f) \circ \alpha_X$ and likewise for g. So

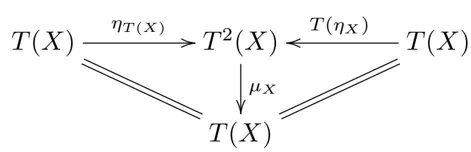
$$F(f) = F(g) \Leftrightarrow \alpha_Y^{-1} \circ G(f) \circ \alpha_X = \alpha_Y^{-1} \circ G(g) \circ \alpha_X \Leftrightarrow G(f) = G(g).$$

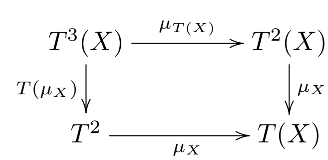
□

0.4.1 Monads

This section recalls the basics of the theory of monads. Some basic references are [83, 10, 88, 17]. Some specific examples are elaborated later on.

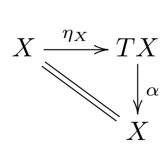
A monad is a functor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations: a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow T$ and a multiplication $\mu : T^2 \Rightarrow T$, such that the following diagrams commute, for $X \in \mathbf{C}$.

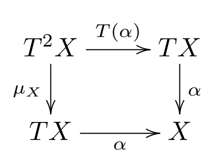


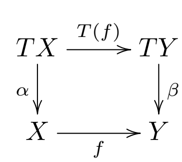


Each adjunction $F \dashv G$ gives rise to a monad $(GF, \eta, G\epsilon F)$.

Given a monad T one can form the category $\mathcal{EM}(T)$ of (Eilenberg-Moore) algebras. Objects of this category are maps of the form $\alpha : T(X) \to X$, making the first two diagrams below commute.







A homomorphism of algebras $(X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ in $\mathcal{C}$ between the underlying objects making the diagram on the above right commute. Therefore the diagram in the middle says that the map α is a homomorphism $(TX, \mu_X) \to (X, \alpha)$. The forgetful functor $U : \mathcal{EM}(T) \to \mathcal{C}$ has a left adjoint, mapping an object $X \in \mathcal{X}$ to the (free) algebra $(T(X), \mu_X)$.

Each category $\mathcal{EM}(T)$ inherits limits from the category $\mathcal{C}$. In the special case where $\mathcal{C} = \mathbf{Set}$, the category of sets and functions, the category $\mathcal{EM}(T)$ is not only complete but also cocomplete (see [10, § 9.3, Prop. 4]).

For any monad $T = (T, \eta, \mu)$ on a category $\mathbf{B}$ we write $\mathcal{K}\ell(T)$ for the Kleisli category of T. Its objects are the same as those of $\mathbf{B}$, but its maps $X \to Y$ are the maps $X \to T(Y)$ in $\mathbf{B}$. The unit $\eta\colon X \to T(X)$ is the identity map $X \to X$ in $\mathcal{K}\ell(T)$; and composition of $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{K}\ell(T)$ is given by $g \odot f = \mu \circ T(g) \circ f$. Maps in such a Kleisli category are understood as computations with outcomes of type T, see [92]. For a monad $T\colon \mathbf{Set} \to \mathbf{Set}$ we write $\mathcal{K}\ell_{\mathbb{N}}(T) \hookrightarrow \mathcal{K}\ell(T)$ for the full subcategory with numbers $n \in \mathbb{N}$ as objects, considered as n-element sets.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, $(T, \eta^T, \mu^T), (S, \eta^S, \mu^S)$ are monads on $\mathcal{C}$ and $\mathcal{D}$ respectively, a *monad functor*[125, §1][3] from S to T is a pair (U, σ) where U is a functor $\mathcal{C} \to \mathcal{D}$ and σ is a natural transformation $SU \to UT$ such that the following diagrams commute

$$
\begin{array}{ccc}
U & \xrightarrow{\eta^S U} & SU \\
& \searrow^{U\eta^T} & \downarrow{\scriptstyle \sigma} \\
& & UT
\end{array}
\qquad
\begin{array}{ccccc}
S^2U & \xrightarrow{S\sigma} & SUT & \xrightarrow{\sigma T} & UT^2 \\
\downarrow{\scriptstyle \mu^S U} & & & & \downarrow{\scriptstyle U\mu^T} \\
SU & & \xrightarrow[\sigma]{} & & UT.
\end{array}
$$

In the case that the $U = \mathrm{Id}$ this is the usual notion of *monad morphism*, see *e.g.* [17, Volume 2, Definition 4.5.8].

Proposition 0.4.6. *For each monad functor* $(U : \mathcal{C} \to \mathcal{D}, \sigma : SU \Rightarrow UT)$ *we can define a functor* $U^\sigma : \mathcal{EM}(T) \to \mathcal{EM}(S)$ *as*

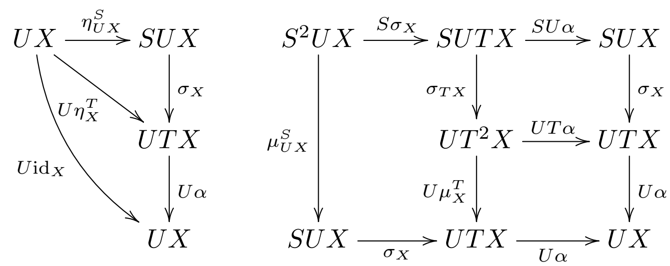
$$
\begin{aligned}
U^\sigma(X, \alpha) &= (UX, U\alpha \circ \sigma_X) \\
U^\sigma(f : (X, \alpha) \to (Y, \beta)) &= U(f),
\end{aligned}
$$

where $(X, \alpha), (Y, \beta)$ *are* T*-algebras,* f *a* T*-algebra map between them.*

Proof. We first show that $U^\sigma(X, \alpha)$ is an S-algebra. We do this by pasting

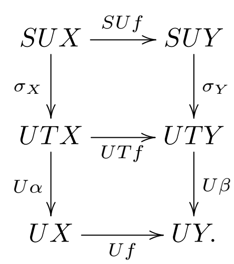
[3] Called a lax map of monads in [84, §6.1].

diagrams as follows:



On the left, the top triangle is part of the definition of monad functor, the bottom one is U applied to the triangle diagram for X being a T-algebra. On the right, the left pentagonal part is from the definition of a monad functor, the upper right square is naturality of σ and the bottom right square is U applied to the square diagram for X being a T-algebra.

We also need to show that if f is a T-algebra map, then $U^\sigma(f)$ is an S-algebra map. We do this by pasting diagrams again:



The top square commutes by naturality of σ, and the bottom square is U applied to the square that commutes because f is a T-algebra map.

Preservation of identities and composition for U^σ follows directly from the fact that U is a functor. □

In the special case of monad morphisms, we can form the category of monads on a given category $\mathcal{C}$, $\mathbf{Monad}(\mathcal{C})$. Given a monad morphism $\sigma : S \Rightarrow T$, we define $\mathcal{EM}(\sigma) = \mathrm{Id}^\sigma : \mathcal{EM}(T) \to \mathcal{EM}(S)$. If we take **Cat** to be the (superlarge) category with large categories as objects and functors as morphism[4] we can prove the following fact.

Proposition 0.4.7. *With the above definition on morphisms, $\mathcal{EM}$ defines a functor* $\mathbf{Monad}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}$.

Proof. We have already seen that $\mathcal{EM}(T)$ is a category for T any monad and that $\mathcal{EM}(f)$ is a functor for any monad morphism (Proposition 0.4.6). Therefore we only need to show that $\mathcal{EM}$ preseves identities and composition.

[4]Not any 2-category and not the large category of small categories.

On objects, $\mathcal{EM}(\mathrm{id}_T)(X,\alpha) = (X, \alpha \circ \mathrm{id}_{TX}) = (X,\alpha)$, and on maps, $\mathcal{EM}(\sigma)$ for any monad morphism σ is equal to the identity functor by the definition of monad morphism as a special monad functor, so $\mathcal{EM}(\mathrm{id}_T) = \mathrm{Id}_{\mathcal{EM}(T)}$.

For composition, let $\sigma : R \Rightarrow S$ and $\tau : S \to T$ be monad morphisms. On objects, we have that

$$\begin{aligned}\mathcal{EM}(\tau)(\mathcal{EM}(\sigma)(X,\alpha)) &= \mathcal{EM}(\tau)(X, \alpha\circ\sigma)\\ &= (X, \alpha\circ\sigma\circ\tau)\\ &= \mathcal{EM}(\sigma\circ\tau)(X,\alpha).\end{aligned}$$

On maps we have that all three functors are the identity, hence agree. Therefore $\mathcal{EM}(\sigma\circ\tau) = \mathcal{EM}(\tau)\circ\mathcal{EM}(\sigma)$ as functors, completing the proof that $\mathcal{EM}$ is itself a functor $\mathbf{Monad}(\mathcal{C}) \to \mathbf{Cat}$. □

The previous proposition shows that isomorphic monads have isomorphic Eilenberg-Moore categories.

We have already seen that adjoints, if they exist, are unique up to natural isomorphism (Proposition 0.4.2). Here we need a stronger result, namely that there is also a monad isomorphism between the induced monads.

Lemma 0.4.8. *Consider a functor $G\colon \mathbf{C} \to \mathbf{D}$ with two left adjoints: $F \dashv G$ and $F' \dashv G$. The induced isomorphism $F \cong F'$ also yields an isomorphism $GF \cong GF'$ of monads on $\mathbf{D}$.*

Proof. Let's write η, ε for the unit and counit of the adjunction $F \dashv G$, and similarly η', ε' for $F' \dashv G$. The multiplication maps for the induced monads GF and GF' are then given by $\mu_X = G(\varepsilon_{FX})\colon GFGF(X) \to GF(X)$ and $\mu'_X = G(\varepsilon'_{F'X})$. There is then a natural isomorphism $\sigma\colon F \Rightarrow F'$ with components:

$$\sigma_X = \Big(F(X) \xrightarrow{F(\eta'_X)} FGF'(X) \xrightarrow{\varepsilon_{F'X}} F'(X) \Big)$$

Then $G\sigma\colon GF \Rightarrow GF'$ is a isomorphism of monads. By using the triangle identities we get:

$$\begin{array}{rcl} G\sigma\circ\eta &=& G(\varepsilon F')\circ FG(\eta')\circ\eta\\ &=& G(\varepsilon F')\circ \eta GF'\circ\eta'\\ &=& \eta'\\ \mu'\circ G\sigma GF'\circ GFG\sigma &=& G\varepsilon' F\circ G\varepsilon F' GF'\circ GF\eta' GF'\circ GFG\varepsilon F'\circ GFGF\eta'\\ &=& G\varepsilon F'\circ GFG\varepsilon' F'\circ GF\eta' GF'\circ GFG\varepsilon F'\circ GFGF\eta'\\ &=& G\varepsilon F'\circ GF(G\varepsilon'\circ\eta' G)F'\circ GFG\varepsilon F'\circ GFGF\eta'\\ &=& G\varepsilon F'\circ GFG\varepsilon F'\circ GFGF\eta'\\ &=& G\varepsilon F'\circ GF\eta'\circ G\varepsilon F\\ &=& G\sigma\circ\mu. \end{array}$$

□

The Distribution Monad

We shall write $\mathcal{D}$ for the discrete probability distribution monad on **Set**. It maps a set X to the set of formal convex combinations $r_1x_1 + \cdots + r_nx_n$, where $x_i \in X$ and $r_i \in [0,1]$ with $\sum_i r_1 = 1$. Alternatively,

$$\mathcal{D}(X) \quad = \quad \left\{\varphi\colon X \to [0,1] \mid \mathrm{supp}(\varphi) \text{ is finite, and } \sum_{x\in X} \varphi(x) = 1\right\},$$

where $\mathrm{supp}(\varphi) \subseteq X$ is the support of φ, containing all x with $\varphi(x) \neq 0$.

We also write $\mathcal{D}_\infty$ for the infinite distribution monad defined as

$$\mathcal{D}_\infty(X) = \left\{\phi : X \to [0,1] \mid \sum_{x\in X} \phi(x) = 1\right\}.$$

It follows from Lemma 0.1.9 that $\mathrm{supp}(\phi)$ is always countable. The functors $\mathcal{D}, \mathcal{D}_\infty : \mathbf{Set} \to \mathbf{Set}$ form monads with the Dirac δ function as the unit:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{D}X \\ x & \longmapsto & \delta_x = y \mapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \end{array} \qquad \begin{array}{ccc} \mathcal{D}\mathcal{D}X & \xrightarrow{\mu_X} & \mathcal{D}X \\ \Psi & \longmapsto & \left(y \mapsto \sum_{\varphi\in\mathcal{D}X} \Psi(\varphi)\cdot\varphi(y)\right). \end{array}$$

This monad is well-known and often occurs in the literature without attribution. Objects of the category $\mathcal{EM}(\mathcal{D})$ of (Eilenberg-Moore) algebras of this monad $\mathcal{D}$ can be considered to be *abstract convex sets*, in interpreting the map $\alpha : \mathcal{D}(X) \to X$ as taking a convex combination $\sum_i r_ix_i$. Morphisms then correspond to affine functions, preserving such convex sums, see [56]. In this thesis we also need to refer to convex subsets of vector spaces, so we have avoided using the term "convex set" for an object of $\mathcal{EM}(\mathcal{D})$. The earliest relation of $\mathcal{D}$ to convex sets we could find in the literature is [127, 4.1.1], where $\mathcal{D}$ is called G, Eilenberg-Moore algebras are called semiconvex sets, and the maps are called semiaffine maps.

The prime example of an Eilenberg-Moore algebra of $\mathcal{D}$ is the unit interval $[0,1] \subseteq \mathbb{R}$ of probabilities. Also, for an arbitrary set X, the set of functions $[0,1]^X$, or fuzzy predicates on X, is a convex set, via pointwise convex sums.

The Ultrafilter Monad

A particular monad that plays an important role later in the thesis is the ultrafilter monad $\mathcal{U}\colon \mathbf{Set} \to \mathbf{Set}$, given by:

$$\begin{array}{rcl} \mathcal{U}(X) & = & \{\mathcal{F} \subseteq \mathcal{P}(X) \mid \mathcal{F} \text{ is an ultrafilter}\} \\ & \cong & \{f\colon \mathcal{P}(X) \to \{0,1\} \mid f \text{ is a homomorphism of Boolean algebras}\} \end{array} \tag{4}$$

where $\{0,1\}$ is the 2-element Boolean algebra. Such an ultrafilter $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfies, by definition, the following three properties.

- It is an up-set: $V \supseteq U \in \mathcal{F} \Rightarrow V \in \mathcal{F}$;
- It is closed under finite intersections: $X \in \mathcal{F}$ and $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$;
- For each set U either $U \in \mathcal{F}$ or $\neg U = \{x \in X \mid x \notin U\} \in \mathcal{F}$, but not both. As a consequence, $\emptyset \notin \mathcal{F}$.

For a function $f\colon X \to Y$ one obtains $\mathcal{U}(f)\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ by:

$$\mathcal{U}(f)(\mathcal{F}) \quad = \quad \{V \subseteq Y \mid f^{-1}(V) \in \mathcal{F}\}.$$

Taking ultrafilters is a monad, with unit $\eta\colon X \to \mathcal{U}(X)$ given by principal ultrafilters:

$$\eta(x) \quad = \quad \{U \subseteq X \mid x \in U\}.$$

The multiplication $\mu\colon \mathcal{U}^2(X) \to \mathcal{U}(X)$ is:

$$\mu(\mathcal{A}) \quad = \quad \{U \subseteq X \mid D(U) \in \mathcal{A}\} \qquad \text{where} \quad D(U) \quad = \quad \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}.$$

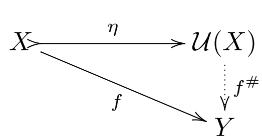
The set $\mathcal{U}(X)$ of ultrafilters on a set X is a topological space with basic (compact) clopens given by subsets $D(U) = \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}$, for $U \subseteq X$. This makes $\mathcal{U}(X)$ into a compact Hausdorff space. The unit $\eta\colon X \to \mathcal{U}(X)$ is a dense embedding.

In fact, $\mathcal{U}(X)$, with this compact Hausdorff topology, defines a left adjoint to the forgetful functor **CHaus** $\to$ **Set**, where **CHaus** is the category of compact Hausdorff spaces and continuous maps, the full subcategory of **Top**, the category of topological spaces and continuous maps.

The following result, that this adjunction is monadic, shows the importance of the ultrafilter monad, see *e.g.* [87], [83, VI.9], [68, III.2], or [17, Vol. 2, Prop. 4.6.6].

Theorem 0.4.9 (Manes)**.** *$\mathcal{EM}(\mathcal{U}) \simeq$ **CHaus**, i.e. the category of algebras of the ultrafilter monad is equivalent to the category **CHaus** of compact Hausdorff spaces and continuous maps.*

The proof is complicated and will not be reproduced here. We only extract the basic constructions. For a compact Hausdorff space Y one uses denseness of the unit η to define a unique continuous extensions $f^{\#}$ as in:



(5)

One defines $f^{\#}(\mathcal{F})$ to be the unique element in $\bigcap\{\overline{V} \mid V \subseteq Y$ with $f^{-1}(V) \in \mathcal{F}\}$. This intersection is a singleton precisely because Y is a compact Hausdorff space. In such a way one obtains an algebra $\mathcal{U}(Y) \to Y$ as extension of the identity.

Conversely, given an algebra $\mathrm{ch}_X\colon \mathcal{U}(X) \to X$ one defines $U \subseteq X$ to be closed if for all $\mathcal{F} \in \mathcal{U}(X)$, $U \in \mathcal{F}$ implies $\mathrm{ch}(\mathcal{F}) \in U$. This yields a topology on X which is Hausdorff and compact. There can be at most one such algebra structure $\mathrm{ch}_X : \mathcal{U}(X) \to X$ on a set X corresponding to a compact Hausdorff topology, because of the following standard result.

Lemma 0.4.10. *Let X be a set with two topologies $\mathcal{O}_1(X), \mathcal{O}_2(X) \subseteq \mathcal{P}(X)$ with $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, $\mathcal{O}_1(X)$ is Hausdorff and $\mathcal{O}_2(X)$ is compact, then $\mathcal{O}_1(X) = \mathcal{O}_2(X)$.* □

Proof. If U is closed in $\mathcal{O}_2(X)$, then it is compact, and, because $\mathcal{O}_1(X) \subseteq \mathcal{O}_2(X)$, also compact in $\mathcal{O}_1(X)$. Hence it is closed there. □

We can apply this result to the space $\mathcal{U}(X)$ of ultrafilters: as described before Theorem 0.4.9, $\mathcal{U}(X)$ carries a compact Hausdorff topology with base $D(U) = \{\mathcal{F} \in \mathcal{U}(X) \mid U \in \mathcal{F}\}$ of clopens. Since it is a free $\mathcal{U}$-algebra by the map $\mu_X : \mathcal{U}^2(X) \to \mathcal{U}(X)$, it has compact Hausdorff topology by Manes's theorem. It is not hard to see that the subsets $D(U)$ are closed in the latter topology, so the two topologies on $\mathcal{U}(X)$ coincide by Lemma 0.4.10.

Example 0.4.11. *The unit interval $[0,1] \subseteq \mathbb{R}$ is a standard example of a compact Hausdorff space. Its Eilenberg-Moore algebra $\mathrm{ch} : \mathcal{U}([0,1]) \to [0,1]$ can be described concretely on $\mathcal{F} \in \mathcal{U}([0,1])$ as:*

$$\mathrm{ch}(\mathcal{F}) \quad = \quad \inf\{s \in [0,1] \mid [0,s] \in \mathcal{F}\}. \tag{6}$$

For the proof, recall that $\mathrm{ch}(\mathcal{F})$ is the sole element of the intersection $\bigcap\{\overline{V} \mid V \in \mathcal{F}\}$. Hence if $[0,s] \in \mathcal{F}$, then $\mathrm{ch}(\mathcal{F}) \in \overline{[0,s]} = [0,s]$, so $\mathrm{ch}(\mathcal{F}) \leq s$. This establishes the $(\leq)$-part of (6). Assume next that $\mathrm{ch}(\mathcal{F}) < \inf\{s \mid [0,s] \in \mathcal{F}\}$. Then there is some $r \in [0,1]$ with $\mathrm{ch}(\mathcal{F}) < r < \inf\{s \mid [0,s] \in \mathcal{F}\}$. Then $[0,r]$ is not in $\mathcal{F}$, so that $\neg[0,r] = (r,1] \in \mathcal{F}$. But this means $\mathrm{ch}(\mathcal{F}) \in \overline{(r,1)} = [r,1]$, which is impossible.

Notice that (6) can be strengthened to: $\mathrm{ch}(\mathcal{F}) = \inf\{s \in [0,1] \cap \mathbb{Q} \mid [0,s] \in \mathcal{F}\}$.

Chapter 1

C*-algebras, Probability and Monads

The following chapter originated as the article "From Kleisli Categories to Commutative C-algebras: Probabilistic Gelfand Duality"[44] and its journal version [45]. The introduction to effect modules and the expectation monad comes from [64].*

1.1 Introduction

There are several notions of computation. We have the classical notion of computation, probabilistic computation, where a computer may make random choices, and quantum computation, which uses quantum mechanical interference and measurement. Normally we would consider classical computation to be done on sets, probabilistic computation on some kind of spaces admitting a notion of probability measures , and quantum computation on Hilbert spaces. We can instead use categories with C*-algebras as objects and a choice of either *-homomorphisms (called MIU-map below) or positive unital maps as the morphisms. The general outline is represented in this table.

	set-theoretic	**probabilistic**	**quantum**
C*-algebras	commutative	commutative	non-commutative
maps preserve	multiplication involution unit	positivity unit	positivity unit
maps abbreviation	MIU	PU	PU

We note at this point that positive unital maps coincide with *completely* positive unital maps if either the domain or codomain of a map is a commutative C*-algebra, but not in general. While the quantum case is an important source of motivation, we will deal mainly with the classical and probabilistic cases in this chapter. In particular, we will relate the alternative method of representing probabilistic computation, using monads, to the C*-algebraic approach.

In recent years the methods and tools of category theory have been applied to Hilbert spaces — see *e.g.* [1] and the references there — and also to *C**-algebras, see for instance [101, 91]. In this chapter we relate the distinction between different types of homomorphisms of C*-algebras to the distinction between different types of computation. Moreover, we demonstrate the relevance of monads (and their Kleisli and Eilenberg-Moore categories) in this field. The aforementioned paper [101] concerns itself with only the *-homomorphisms (*a.k.a.* the MIU-maps).

The main results of this chapter can be summarized as follows. The well-known finite ('baby') version of Gelfand duality involves an equivalence between the category of finite sets (and all functions between them), and the opposite of the category of finite-dimensional commutative C*-algebras, with MIU-maps (*-homomorphisms) between them. Diagrammatically:

$$\mathbf{FinSet} \xrightarrow{\simeq} \mathbf{FdCC^*Alg}^{\mathrm{op}}$$

Our first observation is that if we generalize from MIU to PU (positive unital) maps we get an equivalence:

$$\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \xrightarrow{\simeq} \mathbf{FdCC^*Alg}^{\mathrm{op}}_{\mathrm{PU}}$$

where $\mathcal{D}$ is the distribution monad on **Set**, and $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$ is the Kleisli category of this monad, but with objects restricted to natural numbers. This shows that the category $\mathbf{FdCC^*Alg}_{\mathrm{PU}}$ is equivalent to the Lawvere theory of the distribution monad. The details are in Section 1.4.

The main contribution of the paper lies in a generalization of the latter equivalence beyond the finite case, which can be summarized in a diagram:

$$\begin{array}{ccc} \mathbf{CHaus} \;(\circlearrowleft \mathcal{R}) & \xrightarrow[\text{Gelfand}]{\simeq} & \mathbf{CC^*Alg}^{\mathrm{op}} \\ \downarrow \dashv \uparrow & & \downarrow \\ \mathcal{K}\ell(\mathcal{R}) & \xrightarrow[\text{new}]{\simeq} & \mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}} \end{array} \tag{1.1}$$

At the top of this diagram we have the classical Gelfand duality between the category **CHaus** of compact Hausdorff spaces and the (opposite of the) category of commutative C*-algebras with MIU-maps. Again, the generalisation to the computationally more interesting PU-maps involves a duality with a Kleisli category, namely the Kleisli category $\mathcal{K}\ell(\mathcal{R})$ of what we call the Radon monad $\mathcal{R}$ on compact Hausdorff spaces. By the Riesz representation theorem, elements of $\mathcal{R}(X)$ can be described as Radon probability measures, which in this case coincide with inner regular probability measures (see [114, Theorem 2.14]). We will see more of this later (Theorem 5.4.10).

The closest result in the literature to (1.1) in [80, §2] that relates Kleisli maps for the Giry monad $\mathcal{G}$ of the form $X \to \mathcal{G}(Y)$[1], and maps $\mathcal{L}^\infty(Y) \to \mathcal{L}^\infty(X)$.

Incidentally, the adjunction on the left in Diagram (1.1) can be transferred to the right, and then yields a right adjoint to the inclusion $\mathbf{CC^*Alg} \hookrightarrow \mathbf{CC^*Alg}_{\mathrm{PU}}$. In [132] it is shown that such a right adjoint also exists in the general non-commutative case.

Giry [47, I.4] described how we can consider a stochastic process as being a diagram in the Kleisli category of the Giry monad on measure spaces. By using the Radon monad $\mathcal{R}$ on compact spaces instead, we can get a different category of stochastic processes on compact spaces as diagrams in the (opposite of the) category of *commutative* C*-algebras with PU-maps. This suggests the quantum generalization, considering diagrams in the category of non-commutative C*-algebras. The use of the Kleisli category of $\mathcal{R}$ also suggests that one could generalize to Eilenberg-Moore algebras of $\mathcal{R}$. We will see in chapters 3 and 4 how these two ideas are related.

We also show that the category of C*-algebras with PU-maps embeds fully and faithfully in the category of effect modules, an algebraic structure for predicates adapted to quantum mechanics. At the end of the chapter, we then show, by considering monad morphisms, that Eilenberg-Moore algebras of $\mathcal{R}$ and $\mathcal{E}$ are compact Hausdorff spaces admitting an abstract convex structure.

1.2 Preliminaries on C*-algebras

We write $\mathbf{Vect} = \mathbf{Vect}_{\mathbb{C}}$ for the category of vector spaces over the complex numbers $\mathbb{C}$. This category has direct product $V \oplus W$, forming a biproduct (both a product and a coproduct) and tensors $V \otimes W$, which distribute over $\oplus$. The tensor unit is the space $\mathbb{C}$ of complex numbers. The unit for $\oplus$ is the singleton (null) space 0. We write $\overline{V}$ for the vector space with the same vectors/elements as V, but with conjugate scalar product: $z \bullet_{\overline{V}} v = \overline{z} \bullet_V v$. This makes $\mathbf{Vect}$ an involutive category, see [57].

A **-algebra* is an involutive monoid A in the category $\mathbf{Vect}$. Thus, A is itself a vector space, carries a multiplication $\cdot \colon A \otimes A \to A$, linear in each argument, and has a unit $1 \in A$. Moreover, there is an involution map $(-)^* \colon \overline{A} \to A$, preserving 0 and + and satisfying:

$$\begin{array}{rclcrcl} 1^* & = & 1 & \qquad & (x \cdot y)^* & = & y^* \cdot x^* \\ x^{**} & = & x & & (z \bullet x)^* & = & \overline{z} \bullet x^*. \end{array}$$

Here we have written a fat dot $\bullet$ for scalar multiplication, to distinguish it from the algebra's multiplication $\cdot$. For $z = a + bi \in \mathbb{C}$ we have the conjugate

[1] The article actually uses *Markov kernels*, but these are equivalent to Kleisli morphisms of $\mathcal{G}$ [99].

$\overline{z} = a - bi$. Often we omit the multiplication dot $\cdot$ and simply write xy for $x \cdot y$. Similarly, the scalar multiplication $\bullet$ is often omitted.

An element a of a *-algebra A is called *self-adjoint* if $a^* = a$. The $\mathbb{R}$-linearity of the $-^*$ operation shows that self-adjoint elements form a real subspace of A. If a, b are self-adjoint, then ab is self-adjoint iff $ab = ba$ because $(ab)^* = b^*a^*$.

A *C*-algebra* is a *-algebra A with a norm $\| - \| \colon A \to \mathbb{R}_{\geq 0}$ in which it is complete, satisfying the conditions $\|x\| = 0$ iff $x = 0$ and:

$$\begin{array}{rclcrcl} \|x + y\| & \leq & \|x\| + \|y\| & \qquad & \|z \bullet x\| & = & |z| \cdot \|x\| \\ \|x \cdot y\| & \leq & \|x\| \cdot \|y\| & \qquad & \|x^* \cdot x\| & = & \|x\|^2. \end{array}$$

The last equation $\|x^* \cdot x\| = \|x\|^2$, is the *C*-identity* and distinguishes C*-algebras from Banach *-algebras. We remark at this point that a Banach *-algebra admits at most one norm satisfying the C*-identity. The reason for this is that the spectral radius $r(x)$ is definable in terms of the ring structure of the algebra, and for self-adjoint elements $r(x) = \|x\|$ [69, Proposition 4.1.1 (a)]. If x is an arbitrary element, $x^* \cdot x$ is self-adjoint, so $r(x^* \cdot x) = \|x^* \cdot x\| = \|x\|^2$. In the current setting, each C*-algebra is unital, *i.e.* has a (multiplicative) unit 1. A consequence of the axioms above is that $\|1\| = 1$ unless the C*-algebra is the unique one in which $0 = 1$. A C*-algebra is called *commutative* if its multiplication is commutative, and *finite-dimensional* is it has finite dimension as a vector space.

An element x in a C*-algebra A is called *positive* if it can be expressed as $x = y^* \cdot y$. We write $A^+ \subseteq A$ for the subset of positive elements in A. This subset is a cone, which is to say it is closed under addition and scalar multiplication with positive real numbers, and $A_+ \cap -A_+ = \{0\}$ [29, Proposition 1.6.1]. Positive elements are self-adjoint, and we can deduce from this that the product of two positive elements is positive iff they commute. The square $x^2 = x \cdot x$ of a self-adjoint element $x = x^*$ is obviously positive. The positive cone defines an order on every C*-algebra by (1), this is the usual order on a C*-algebra.

We will consider two options when it comes to maps between C*-algebras. The difference between them plays an important role in this chapter.

Definition 1.2.1. *We define three categories* $\mathbf{C^*Alg}$, $\mathbf{C^*Alg}_{\mathrm{PU}}$ *and* $\mathbf{C^*Alg}_{\mathrm{P}\leq 1}$ *with C*-algebras as objects, but with different morphisms. We also define their full subcategories* $\mathbf{CC^*Alg}$, $\mathbf{CC^*Alg}_{\mathrm{PU}}$ *and* $\mathbf{CC^*Alg}_{\mathrm{P}\leq 1}$ *on commutative C*-algebras.*

(i) A morphism $f \colon A \to B$ *in* $\mathbf{C^*Alg}$ *is a linear map preserving multiplication (M), involution (I), and unit (U). Explicitly, this means for all* $x, y \in A$,

$$f(x \cdot y) \;=\; f(x) \cdot f(y) \qquad f(x^*) \;=\; f(x)^* \qquad f(1) \;=\; 1.$$

*Such "MIU" maps are usually called *-homomorphisms.*

(ii) A morphism $f \colon A \to B$ *in* $\mathbf{C^*Alg}_{\mathrm{PU}}$ *is a linear map that preserves positive elements and the unit. This means that* f *restricts to a function* $A^+ \to B^+$. *Alternatively, for each* $x \in A$ *there is a* $y \in B$ *with* $f(x^*x) = y^*y$.

(iii) A morphism $f : A \to B$ in $\mathbf{C^*Alg}_{\mathrm{P}\leq 1}$ is a linear map that preserves positive elements and maps the unit 1_A to some element $\leq 1_B$, necessarily positive. □

For both $X \in \{MIU, PU, P \leq 1$ there are obvious full subcategories of commutative and/or finite-dimensional C-algebras, as described in:*

$$\begin{array}{ccccc} & & \mathbf{CC^*Alg}_X & & \\ & \nearrow & & \searrow & \\ \mathbf{FdCC^*Alg}_X & & & & \mathbf{C^*Alg}_X \\ & \searrow & & \nearrow & \\ & & \mathbf{FdC^*Alg}_X & & \end{array}$$

Clearly, each "MIU" map is also a "PU" map, and every "PU" map is subunital, so that we have inclusions $\mathbf{C^*Alg} \hookrightarrow \mathbf{C^*Alg}_{\mathrm{PU}} \hookrightarrow \mathbf{C^*Alg}_{\mathrm{P}\leq 1}$, and also for the various subcategories. A map that preserves positive elements is called positive itself; and a unit preserving map is called unital. Positive unital maps are the natural notion of morphism between order unit spaces and Riesz spaces.

The special case in which the codomain is $\mathbb{C}$ is important. We define sets of *states* and *multiplicative states* as:

$$\mathrm{Stat}(A) = \mathbf{C^*Alg}_{\mathrm{PU}}(A, \mathbb{C}) \qquad \text{and} \qquad \mathrm{MStat}(A) = \mathbf{C^*Alg}(A, \mathbb{C}).$$

There is also the commonly used notion of completely positive maps, which is a stronger condition than positivity but weaker than being MIU. These maps are important when defining the tensor of C*-algebras as a functor, as the tensor of positive maps need not be positive. They are also widely considered to represent the physically realizable transformations. Positive, but non-completely positive maps of C*-algebras also have their uses, as entanglement witnesses for example [55, theorem 2]. In general, throughout this thesis we put complete positivity to one side, hoping that it can be added later via a general construction, as is sketched in [46, §4]. In this chapter, we mainly consider the commutative case, where positive and completely positive coincide anyway. In fact, since a completely positive unital map is what is known as a channel in quantum information, then Theorem 1.5.1 shows that every channel in Mislove's sense [90] is a channel in the usual sense.

We collect some basic (standard) properties of PU-morphisms between C*-algebras.

Lemma 1.2.2.

(i) *An element a in a C*-algebra A is can be expressed as $a_{\Re} + ia_{\Im}$ with $a_{\Re}$ and $a_{\Im}$ self-adjoint, defined by the formulas*

$$a_{\Re} = \frac{a + a^*}{2} \qquad\qquad a_{\Im} = \frac{a - a^*}{2i}.$$

This decomposition is unique, i.e. *any decomposition into real and imaginary self-adjoint parts is the same, and we have $\|a_{\Re}\|, \|a_{\Im}\| \leq \|a\|$.*

(ii) Any self-adjoint a can be expressed as $a_+ - a_-$ where $a_+, a_- \in A_+$ and $\|a_+\|, \|a_-\| \leq \|a\|$. We can arrange that $a_+ a_- = 0$.

Proof.

(i) We see that $a_\Re$ and $a_\Im$ are self-adjoint. We have

$$a_\Re + ia_\Im = \frac{a+a^*}{2} + i\frac{a-a^*}{2i} = \frac{a+a^*+a-a^*}{2} = \frac{2a}{2} = a.$$

We now show uniqueness. Suppose that $a = b_\Re + b_\Im$ with $b_\Re$ and $b_\Im$ self-adjoint. Then

$$a_\Re = \frac{a+a^*}{2} = \frac{b_\Re + ib_\Im + b_\Re - ib_\Im}{2} = \frac{2b_\Re}{2} = b_\Re.$$

The proof that $a_\Im = b_\Im$ is similar.

For the inequality, we see:

$$\|a_\Re\| = \left\| \frac{a+a^*}{2} \right\| \leq \frac{1}{2}(\|a\| + \|a^*\|) = \|a\|,$$

and the argument for $\|a_\Im\|$ is similar.

(ii) See any of the following references: [69, Proposition 4.2.3 (iii)] [117, Defn. 1.4.3] [29, §1.5.7 and 1.6.5]. □

Lemma 1.2.3. *A PU-map, i.e. a morphism in the category $\mathbf{C^*Alg}_{\mathrm{PU}}$, preserves self-adjointness of elements, commutes with involution $(-)^*$, and preserves the partial order $\leq$ given by (1).*

Moreover, a PU-map f satisfies $\|f(x)\| \leq 4\|x\|$, so that $\|f(x) - f(y)\| \leq 4\|x - y\|$, making f continuous. In fact, this constant can be reduced to 1, i.e. $\|f(x)\| \leq \|x\|$.

*As every unital *-homomorphism is a PU-map, the above facts are also true of all unital *-homomorphisms.*

Proof. Let $f : A \to B$ be a PU map. By Lemma 1.2.2, if a is a self-adjoint element, we have $a = a_+ - a_-$, where a_+, a_- are positive, and so $f(a) = f(a_+) - f(a_-)$, a difference of two positive elements, and therefore a self-adjoint element. If a is a general element, it can be expressed as $a_\Re + ia_\Im$, $a_\Re, a_\Im$ being self adjoint. We therefore have

$$\begin{aligned} f(a^*) &= f((a_\Re + ia_\Im)^*) = f(a_\Re - ia_\Im) = f(a_\Re) - if(a_\Im) \\ &= (f(a_\Re) + if(a_\Im))^* = f(a_\Re + ia_\Im)^* = f(a)^*. \end{aligned}$$

Preservation of the partial order is implied by preservation of positive elements.

For positive a we have $a \leq \|a\| \bullet 1$, and thus $f(a) \leq \|a\| \bullet 1$, which gives $\|f(a)\| \leq \|a\|$. An arbitrary element $a \in A$ can be written as linear combination of four positive elements x_i, as in $x = x_1 - x_2 + ix_3 - ix_4$, with $\|x_i\| \leq \|x\|$.

Finally, $\|f(x)\| = \|f(x_1) - f(x_2) + if(x_3) - if(x_4)\| \leq \sum_i \|f(x_i)\| \leq \sum_i \|x_i\| \leq 4\|x\|$.

The reduction of the constant to 1 follows from the Russo-Dye theorem [115, Corollary 1]. □

We next recall two famous adjunctions involving compact Hausdorff spaces. The first one is due to Manes [87] and describes compact Hausdorff spaces as monadic over **Set**, via the ultrafilter monad (see Theorem 0.4.9). The second one is known as Gelfand duality , relating compact Hausdorff spaces and *commutative* C*-algebras. Notice that this result involves the "MIU" maps.

Theorem 1.2.4. *Let* **CHaus** *be the category of compact Hausdorff spaces, with continuous maps between them. There are two fundamental adjunctions:*

$$\begin{array}{c} \mathbf{CHaus} \\ \mathcal{U}\ \uparrow \ \dashv \ \downarrow\ \mathit{forget} \\ \mathbf{Set} \end{array} \qquad\qquad \begin{array}{c} \mathbf{CHaus} \\ C\ \downarrow \ \simeq \ \uparrow\ \mathrm{Spec} \\ \mathbf{CC^*Alg}^{\mathrm{op}} \end{array}$$

On the left the functor $\mathcal{U}$ sends a set X to the ultrafilters on the powerset $\mathcal{P}(X)$. And on the right the equivalence of categories is given by sending a compact Hausdorff space X to the commutative C-algebra $C(X) = \mathbf{Top}(X, \mathbb{C})$ of continuous functions $X \to \mathbb{C}$. The underlying set of $\mathrm{Spec}(A)$ is $\mathrm{MStat}(A)$, and the topology is the weak-* topology $\sigma(A^*, A)$, as states are elements of A^* by Lemma 1.2.3.*

The unit and counit of Gelfand duality are

$$\begin{aligned} \eta_X &: X \to \mathrm{Spec}(C(X)) \\ \eta_X(x)(a) &= a(x) \\ \epsilon_A &: C(\mathrm{Spec}(A)) \leftarrow A \\ \epsilon_A(a)(\phi) &= \phi(a) \end{aligned}$$

□

The multiplicative states on a commutative C*-algebra can equivalently be described as maximal ideals, or also as pure states (see below).

Corollary 1.2.5. *For each finite-dimensional commutative C*-algebra A there is an $n \in \mathbb{N}$ with $A \cong \mathbb{C}^n$ in* **FdCC*Alg**.

Proof. By the previous theorem there is a compact Hausdorff space X such that A is MIU-isomorphic to the algebra of continuous maps $X \to \mathbb{C}$. This X must be finite, and since a finite Hausdorff space is discrete, all maps $X \to \mathbb{C}$ are continuous. Let $n \in \mathbb{N}$ be the number of elements in X; then we have an isomorphism $A \cong \mathbb{C}^n$. □

As we can already see in the above theorem, it is the *opposite* of a category of C*-algebras that provides the most natural setting for computations. This is in line with what is often called the Heisenberg picture. In a logical setting it corresponds to computation of weakest preconditions, going backwards. The situation may be compared to the category of complete Heyting algebras, which is most usefully known in opposite form, as the category of locales, see [68].

The set of states $\mathrm{Stat}(A) = \mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}(A, \mathbb{C})$ can be equipped with the weak-* topology, which is the coarsest (smallest) topology in which all evaluation maps $\mathrm{ev}_x = \phi \mapsto \phi(x) \colon \mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}(A, \mathbb{C}) \to \mathbb{C}$, for $x \in A$, are continuous. We introduce the category **CCL**, which first appeared in [126], in order to extend Stat to a functor.

The category **CCL** has as its objects compact convex subsets of (Hausdorff) locally convex topological vector spaces. More accurately, the objects are pairs (V, X) where V is a (Hausdorff) locally convex topological vector space, and X is a compact convex subset of V. The maps $(V, X) \to (W, Y)$ are continuous, affine maps $X \to Y$. Note that if (V, X) and (W, Y) are isomorphic, while X is necessarily homeomorphic to Y, V need not bear any particular relation to W at all. We can see **CCL** forms a category, as identity maps are affine and continuous and both of these attributes of a map are preserved under composition. We remark at this point that we have a forgetful functor $U \colon \mathbf{CCL} \to \mathbf{CHaus}$, taking the underlying compact Hausdorff space of X.

Proposition 1.2.6. *For a C*-algebra A, the states* $\mathrm{Stat}(A) = \mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}(A, \mathbb{C})$ *form a convex, compact Hausdorff subspace of the dual space of A given the weak-* topology. Each PU-map* $f \colon A \to B$ *yields an affine continuous function* $\mathrm{Stat}(f) = (-) \circ f \colon \mathrm{Stat}(B) \to \mathrm{Stat}(A)$. *This defines a functor*

$$\mathrm{Stat} \colon \mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}^{\mathrm{op}} \to \mathbf{CCL}.$$

We recall that a function (between convex sets) is called *affine* if it preserves convex sums. As we saw in section 0.4.1 such affine maps are homomorphisms of Eilenberg-Moore algebras for the distribution monad $\mathcal{D}$.

Proof. For each finite collection $h_i \in \mathbf{C}^*\mathbf{Alg}_{\mathrm{PU}}(A, \mathbb{C})$ with $r_i \in [0, 1]$ satisfying $\sum_i r_i = 1$, the function $h = \sum_i r_i h_i$ is again a state. Moreover, such convex sums are preserved by precomposition, making the maps $(-) \circ f$ affine.

The fact that the dual space of A, given the weak-* topology, is a locally convex space is standard (Proposition 0.3.1 and after). This implies that the space of states is Hausdorff. The set of positive linear functionals is defined to be the dual cone of the positive operators, so is closed (Lemmas 0.3.7 and 0.3.5) and the set of linear functionals such that $\phi(1) = 1$ is weak-* closed, and the set of states is the intersection of the two, and therefore closed. The space of states is also bounded as each state has norm 1. Therefore the state space is a closed and bounded and hence compact by the Banach-Alaoglu Theorem.

Precomposition $(-) \circ f$ is continuous, since for $x \in A$ and $U \subseteq \mathbb{C}$ open we get an open subset $\big((-) \circ f\big)^{-1}(\mathrm{ev}_x^{-1}(U)) = \{h \mid \mathrm{ev}_x(h \circ f) \in U\} = \mathrm{ev}_{f(x)}^{-1}(U)$.

Precomposition with the identity map gives the same state again, so Stat preserves identity maps. Since composition of PU-maps is associative, Stat preserves composition, and so is a functor. □

1.2.1 Effect modules

This section introduces effect modules and notions related to them, referring to [56, 62, 63]. Intuitively, effect modules are like vector spaces, but instead of $\mathbb{R}$ as scalars, we have $[0,1]$, and instead of an underlying abelian group, they have an underlying effect algebra. Effect modules were introduced as "convex effect algebras" in [48].

Effect algebras were introduced in mathematical physics, in the investigation of quantum probability, see [38, 33]. An *effect algebra* is a partial commutative monoid $(M, 0, \ovee)$ with an orthocomplement $(-)^\perp$. One writes $x \perp y$ if $x \ovee y$ is defined. Commutativity of $\ovee$ need to be defined in such a way that existence of $a \ovee b$ implies existence of $b \ovee a$, and an analogous condition is also necessary for associativity. The orthocomplement satisfies $x^{\perp\perp} = x$ and $x \ovee x^\perp = 1$, where $1 = 0^\perp$. We also require that $a \ovee b = 1$ implies $b = a^\perp$, uniqueness of orthocomplement. On any effect algebra there is always a partial order, given by $x \leq y$ iff there exists a z such that $x \ovee z = y$. Our main example of an effect algebra is the unit interval $[0,1] \subseteq \mathbb{R}$, where addition $+$ is made partial, $a + b$ being defined only if the sum is in $[0,1]$. This is commutative, associative, and has 0 as a unit; moreover, the orthocomplement is $r^\perp = 1 - r$. We write **EA** for the category of effect algebras, where the morphisms are maps preserving $\ovee$ and 1 — and thus all other structure.

For each set X, the set $[0,1]^X$ of fuzzy predicates on X is an effect algebra, via pointwise operations. Each Boolean algebra B is an effect algebra with $x \perp y$ iff $x \wedge y = \perp$; then $x \ovee y = x \vee y$. In a quantum setting, the motiviating example is the set of effects $\mathrm{Ef}(H) = \{E \colon H \to H \mid 0 \leq E \leq I\}$ on a Hilbert space H, see *e.g.* [33, 51].

The category **EA** carries a symmetric monoidal structure $\otimes$ with the 2-element effect algebra $\{0,1\}$ as tensor unit (which is at the same time the initial object), see [62]. The usual multiplication of real numbers (probabilities in this case) yields a monoid structure on $[0,1]$ in the category **EA**. An *effect module* is then an effect algebra with an $[0,1]$-action $[0,1] \otimes E \to E$. Explicitly, it can be described as a scalar multiplication $(r, x) \mapsto rx$ satisfying:

$$\begin{array}{rclcrcll} 1x & = & x & \qquad & (r+s)x & = & rx + sx & \text{if } r+s \leq 1 \\ (rs)x & = & r(sx) & & r(x \ovee y) & = & rx \ovee ry & \text{if } x \perp y. \end{array}$$

In particular, if $r + s \leq 1$, then a sum $rx \ovee sy$ always exists (Lemma A.4.1, see also [48]).

The algebras $[0,1]^X$ and $\mathrm{Ef}(H)$ are clearly effect modules. Other examples of effect modules, occurring in integration theory, are the sets $[X \to_s [0,1]]$ of *simple* functions $X \to [0,1]$, having only finitely many output values.

A morphism $E \to D$ in the category **EMod** of effect modules is a function $f \colon E \to D$ between the underlying sets satisfying:

$$\begin{array}{rclcrcl} f(rx) & = & rf(x) & \qquad & f(1) & = & 1 \\ f(x \ovee y) & = & f(x) \ovee f(y) & & \text{if} & & x \perp y. \end{array}$$

Each effect module can be "totalized" to produce a partially ordered vector space with a special kind of unit. We first explain totalization of effect algebras, from [62, Proposition 3].

Proposition 1.2.7. *There is a coreflection*

$$\mathbf{EA} \underset{[0,u]_{(-)}}{\overset{\mathcal{T}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{BCM} \tag{1.2}$$

where **BCM** *is the category of "barred commutative monoids": its objects are pairs* (M, u)*, where* M *is a commutative monoid and* $u \in M$ *is a unit such that* $x + y = 0$ *implies* $x = y = 0$ *and* $x + y = x + z = u$ *implies* $y = z$*. The morphisms in* **BCM** *are monoid homomorphisms that preserve the unit. As this is a coreflection every effect algebra* E *is isomorphic to* $[0, u]_{\mathcal{T}(E)}$. □

The partialization functor $[0, u]_{(-)}$ in (1.2) is defined by the 'unit interval':

$$[0, u]_M \quad = \quad \{x \in M \mid x \preceq u\},$$

where $x \preceq y$ iff there exists a z such that $x + z = y$. The operation $\otimes$ is defined by $x \otimes y = x + y$ but this is only defined if $x + y \preceq u$, *i.e.* $x + y \in [0, u]_M$.

The totalization of the effect algebra $\{0, 1\}$ is the natural numbers $\mathbb{N}$ with 1 as the unit, and the totalization of $[0, 1]$ is $\mathbb{R}_{\geq 0}$, again with 1 as the unit. An important difference between these examples is that in $\mathbb{N}$, if we pick different non-zero elements as order units, we obtain non-isomorphic objects in **BCM** and non-isomorphic unit intervals. However, for $\mathbb{R}_{\geq 0}$, the choices of order unit are all isomorphic.

We can now discuss the totalization of effect modules. In an ordered vector space, (A, A_+), $u \in A_+$ is a *strong order unit* if for all $x \in A$, there is some $\alpha \in \mathbb{R}_{\geq 0}$ such that $-\alpha u \leq x \leq \alpha u$. It is equivalent to require that A_+ be generating and that for all $x \in A_+$ there is a λ such that $x \leq \lambda u$, by Lemma A.5.1. A triple (A, A_+, u) where (A, A_+) is an ordered vector space and u a strong order unit is called a *partially ordered vector space with unit* in [63, before Theorem 3], and the category **poVectu** has these as objects and the maps are linear positive maps preserving the unit. We call these maps *(positive) unital maps*, as in the C*-algebraic case.

We define the unit ball of $(A, A_+, u) \in \mathbf{poVectu}$ as $U = [-u, u] = \{x \in A | -u \leq x \leq u\}$. This is absolutely convex and absorbing, so its Minkowski functional $\|\text{-}\|_U$ defines a seminorm on E. We say that (A, A_+, u) is *archimedean* if $x \leq \frac{1}{n} u$ for all $n \in \mathbb{N}_{>0}$ implies $x \in -A_+$. This implies that $\|\text{-}\|_U$ is a norm, in which the positive cone is closed (Lemma A.5.3). Be warned that the condition one might expect, that $x \in A_+$ and $x \leq \frac{1}{n} u$ for all $n \in \mathbb{N}_{>0}$ implies $x = 0$, is strictly weaker than archimedeanness and is known as being almost Archimedean in [66, 1.3.7], and this is equivalent to $\|\text{-}\|_U$ being a norm.

A partially ordered vector space with unit (A, A_+, u) is called an *order-unit space* if it is archimedean. If it is complete in its norm, then it is a *Banach* *order-unit space*. In [63, p.154 and Proposition 11] the categories **OUS** and **BOUS** are defined, having order-unit spaces and Banach order-unit spaces as objects (respectively), and with maps being positive maps that preserve the unit, *i.e.* as full subcategories of **poVectu**. We also define $\mathbf{OUS}_{\leq 1}$ and $\mathbf{BOUS}_{\leq 1}$, the category of order-unit spaces and *subunital maps*, which are maps $f : (E, E_+, u) \to (F, F_+, v)$ such that $f(u) \leq v$, and its full subcategory on Banach order-unit spaces.

We note at this point that we allow $(\{0\}, \{0\}, 0)$ as an order-unit space.

Proposition 1.2.8. *If $f : (E, E_+, u) \to (F, F_+, v)$ is subunital or unital, $\|f\| \leq 1$. If $F \neq 0$ and f is unital, $\|f\| = 1$.*

Proof. To show that $\|f\| \leq 1$, it is sufficient to show that if U is the closed unit ball of E and V is the closed unit ball of F, $f(U) \subseteq V$. Since $U = [-u, u]$ and $V = [-v, v]$, all we need to show is that if $-u \leq x \leq u$, then $-v \leq f(x) \leq v$. Whether the map is taken to be unital or subunital, we have $f(u) \leq v$. By the positivity and linearity of f, we have

$$-v \leq f(-u) \leq f(x) \leq f(u) \leq v.$$

Now assume that $F \neq 0$. By Lemma A.5.2, $v \neq 0$ and $\|v\| = 1$. Since $f(u) = v$, we must also have $u \neq 0$, or we would have $v = 0$ by linearity, and so $\|u\| = 1$ as well. Since $f(u) = v$, f maps an element of norm 1 to an element of norm 1, so has operator norm at least 1. Since $\|f\| \leq 1$, we have $\|f\| = 1$. □

A particular consequence of the above is that every map, unital or subunital, of order-unit spaces is continuous, and isomorphisms between order-unit spaces are isometries of the underlying Banach spaces.

There are also full subcategories of **EMod** on *archimedean effect modules* and *Banach effect modules*, **AEMod** and **BEMod** respectively. These are defined in [63, pp. 154-155]. An effect module is archimedean if $x \leq y$ holds when $\frac{1}{2}x \leq \frac{1}{2}y \ovee \frac{r}{2}$ for all $r \in (0, 1]$, and a metric can be defined [63, (10)] on each archimedean effect module, and Banach effect modules are those that are complete in this metric.

Theorem 1.2.9. *The unit interval functor $[0, 1]_{-}$: **poVectu** $\to$ **EMod** is an equivalence of categories, with $\mathcal{T}$: **EMod** $\to$ **poVectu**. Restricting these functors gives adjoint equivalences **OUS** $\simeq$ **AEMod** and **BOUS** $\simeq$ **BEMod**.*

Proof. See [63, Theorem 3] and [63, Propositions 9,11]. □

We review our examples of effect modules: both the effect modules $[0, 1]$ and $[0, 1]^X$ are Archimedean, and also Banach effect modules. Norms and distances in $[0, 1]$ are the usual ones, but limits in $[0, 1]^X$ are defined via the supremum (or uniform) norm.

For any C*-algebra A, we can define $\mathrm{SA}(A)$ to be the set of self-adjoint elements. This is an $\mathbb{R}$-subspace of A, and it is closed because it is equal to $(\mathrm{id}_A - \text{-}^*)^{-1}(\{0\})$, the preimage of a closed set under a continuous map. The positive cone $A_+ \subseteq \mathrm{SA}(A)$, and in fact $\mathrm{SA}(A) = A_+ - A_+$ [29, §1.5.7 and 1.6.5], so $(\mathrm{SA}(A), A_+)$ is a directed ordered vector space. The reader can probably see where this is going:

Proposition 1.2.10. *For each C*-algebra A,* $(\mathrm{SA}(A), A_+, 1_A)$ *is a Banach order-unit space. If, for any PU-map* $f : A \to B$ *we define* $\mathrm{SA}(f) = f|_{\mathrm{SA}(A)}$*, then* SA *is a functor* $\mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{BOUS}$*, and similarly for subunital maps we get a functor* $\mathbf{C^*Alg}_{\mathrm{P}\leq 1} \to \mathbf{BOUS}_{\leq 1}$*. These functors are full and faithful.*

Proof. For the proof that $(\mathrm{SA}(A), A_+, 1_A)$ is a Banach order-unit space, see [34, Proposition 5.2] or [6, Theorem 1.95], although undoubtedly the definition of order-unit space was motivated by $\mathrm{SA}(A)$ in the first place. As positive maps preserve self-adjoint elements (Lemma 1.2.3), the map $\mathrm{SA}(f)$ is well-defined, and its linearity, positivity and preservation of unit follow directly. Preservation of identities and composition by SA is trivial.

To show that SA is faithful, let $f, g : A \to B$ be maps in $\mathbf{C^*Alg}_{\mathrm{PU}}$ or $\mathbf{C^*Alg}_{\mathrm{P}\leq 1}$ (the proof is the same in either case) and suppose $\mathrm{SA}(f) = \mathrm{SA}(g)$. Then for any $a \in A$, applying Lemma 1.2.2 to express $a = a_{\Re} + ia_{\Im}$, we observe

$$f(a) = f(a_{\Re}+ia_{\Im}) = \mathrm{SA}(f)(a_{\Re})+i\mathrm{SA}(f)(a_{\Im}) = \mathrm{SA}(g)(a_{\Re})+i\mathrm{SA}(g)(a_{\Im}) = g(a),$$

so $f = g$.

To show that SA is full, let $f : \mathrm{SA}(A) \to \mathrm{SA}(B)$ be a positive subunital map. Using Lemma 1.2.2 again, define

$$g(a) = f(a_{\Re}) + if(a_{\Im})$$

for all $a \in A$.

To prove the additive part of linearity, consider $a+b$. We have that $a_{\Re} + b_{\Re}$ is self-adjoint, as is the sum of the imaginary parts, and so by the uniqueness of the decomposition (Lemma 1.2.2) $(a + b)_{\Re} = a_{\Re} + b_{\Re}$ and $(a + b)_{\Im} = a_{\Im} + b_{\Im}$. So

$$\begin{aligned} g(a + b) &= f(a_{\Re} + b_{\Re}) + if(a_{\Im} + b_{\Im}) = f(a_{\Re}) + f(b_{\Re}) + if(a_{\Im}) + if(b_{\Im}) \\ &= g(a) + g(b). \end{aligned}$$

For the multiplicative part of linearity, we first show it for multiplication by a real. Let $\alpha \in \mathbb{R}$. Since $\alpha a_{\Re}$ and $\alpha a_{\Im}$ are self-adjoint, they are the real and imaginary parts of αa, so

$$g(\alpha a) = f(\alpha a_{\Re}) + if(\alpha a_{\Im}) = \alpha f(a_{\Re}) + i\alpha f(a_{\Im}) = \alpha g(a).$$

Now we show that g preserves multiplication by i. We see that $(ia)_{\Re} = -a_{\Im}$ and $(ia)_{\Im} = a_{\Re}$, so

$$\begin{aligned} g(ia) &= f((ia)_{\Re}) + if((ia)_{\Im}) = f(-a_{\Im}) + if(a_{\Re}) = if(a_{\Re}) + i(if(a_{\Im})) \\ &= i(f(a_{\Re}) + if(a_{\Im})) = ig(a). \end{aligned}$$

We can now prove $\mathbb{C}$-linearity. Take $z = \alpha + i\beta$. Then

$$g(za) = g(\alpha a + i\beta a) = g(\alpha a) + g(i\beta a) = \alpha g(a) + i\beta g(a) = zg(a).$$

We have that $g(1_A) = f(1_A) \leq 1_B$ by subunitality of f. If f is unital, then $f(1_A) = 1_B$, so g is unital. We also have that if a is positive, then its imaginary part is 0, so $g(a) = f(a)$, which is positive since f is a positive map. Thus we have fullness in both cases. □

For a C*-algebra A we can define write $[0,1]_A \subseteq A^+ \subseteq A$ for the subset of positive elements below the unit. We see immediately that $[0,1]_A = [0,1]_{\mathrm{SA}(A)}$. The elements in $[0,1]_A$ are known as *effects* (or sometimes also as *predicates*). This extends the definition of effect we saw before for $B(\mathcal{H})$, and we shall see in a moment that, in fact, it unifies this example with the other example $[0,1]^X$, which is $[0,1]_{\ell^\infty(X)}$.

Each PU-map of C*-algebras $f : A \to B$ preserves $\leq$ and thus restricts to $[0,1]_A \to [0,1]_B$. This restriction is a map of effect modules. Hence we get a "predicate" functor $\mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{EMod}$. This map is equal to $[0,1]_{\mathrm{SA}(f)}$. Therefore we have

Corollary 1.2.11. *The functor* $[0,1]_{(-)} \colon \mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{BEMod}$ *is full and faithful.*

Proof. $[0,1]_{-} : \mathbf{BOUS} \to \mathbf{BEMod}$ is an equivalence by Theorem 1.2.9, and therefore full and faithful, and $\mathrm{SA} : \mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{BOUS}$ is full and faithful by Proposition 1.2.10. Therefore their composite $[0,1]_{-} : \mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{BEMod}$ is full and faithful. □

1.3 Set-theoretic Computations in C*-algebras

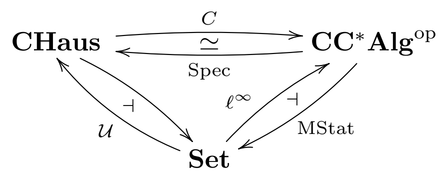
For a set X, a function $f \colon X \to \mathbb{C}$ is called *bounded* if $|f(x)| \leq s$, for some $s \in \mathbb{R}_{\geq 0}$. We write $\ell^\infty(X)$ for the set of such bounded functions. Notice that if X is finite, any function $X \to \mathbb{C}$ is bounded, so that $\ell^\infty(X) = \mathbb{C}^X$.

Each $\ell^\infty(X)$ is a commutative C*-algebra, with pointwise addition, multiplication and involution, and with the uniform/supremum norm:

$$\|f\|_\infty \quad = \quad \inf\{s \in \mathbb{R}_{\geq 0} \mid \forall x.\, |f(x)| \leq s\}.$$

In fact it is a typical example of a commutative W*-algebra, but we will leave W*-algebras to subsequent chapters. This yields a functor $\ell^\infty \colon \mathbf{Set} \to \mathbf{CC^*Alg}^{\mathrm{op}}$, where for $h \colon X \to Y$ we have $\ell^\infty(h) = (-) \circ h \colon \ell^\infty(Y) \to \ell^\infty(X)$; it preserves the (pointwise) operations. We have the following result.

Proposition 1.3.1. *The functor* $\ell^\infty \colon \mathbf{Set} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ *is left adjoint to the multiplicative states functor* $\mathrm{MStat} : \mathbf{CC^*Alg}^{\mathrm{op}} \to \mathbf{Set}$*. In combination with* *the adjunctions from Theorem 1.2.4 we get a situation:*



By composition and uniqueness of adjoints we get:

$$C \circ \mathcal{U} \;\cong\; \ell^\infty \qquad \textit{and also} \quad \mathrm{Spec} \circ \ell^\infty \;\cong\; \mathcal{U}.$$

Proof. Recall that MStat is the set underlying the compact Hausdorff space Spec. We first show $\ell^\infty \dashv \mathrm{MStat}$ using by defining the unit and verifying the universal property (Theorem 0.4.1 (i)). We define the unit $\eta_X : X \to \mathrm{MStat}(\ell^\infty(X))$, where $X \in \mathbf{Set}$, as

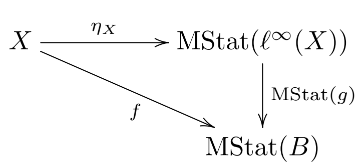
$$\eta_X(x)(a) \;=\; a(x),$$

where $a \in \ell^\infty(X)$. Then $\eta_X(x)$ is a multiplicative state on $\ell^\infty(X)$ because the vector space structure, multiplication and multiplicative unit are defined pointwise. To show the naturality square for η commutes, we must show that for all $f : X \to Y$ in $\mathbf{Set}$, $\mathrm{MStat}(\ell^\infty(f)) \circ \eta_X = \eta_Y \circ f$. If we take $x \in X$ and $b \in \ell^\infty(Y)$, we have:

$$\begin{aligned}
\big(\mathrm{MStat}(\ell^\infty(f)) \circ \eta_X\big)(x)(b) &= \mathrm{MStat}(\ell^\infty(f))(\eta_X(x))(b) \\
&= (\eta_X(x) \circ \ell^\infty(f))(b) \\
&= \eta_X(x)(\ell^\infty(f)(b)) \\
&= \eta_X(x)(b \circ f) \\
&= b(f(x)) \\
&= \eta_Y(f(x))(b) \\
&= (\eta_Y \circ f)(x)(b).
\end{aligned}$$

We now show this natural transformation satisfies the universal property making it the unit of the adjunction. Let $X \in \mathbf{Set}$, $B \in \mathbf{CC^*Alg}$ and $f : X \to \mathrm{MStat}(B)$. Define $g : B \to \ell^\infty(X)$ as $g(b)(x) = f(x)(b)$. We must show that $g(b)$ is an element of $\ell^\infty(X)$, *i.e.* that it is bounded. For all $x \in X$, $f(x)$ is a multiplicative state, hence a state, so by [29, Proposition 2.1.4] we have $\|f(x)\| = 1$, and so $|g(b)(x) = |f(x)(b)| \leq \|f(x)\|\|b\| = \|b\|$. Therefore $\|b\|$ is a bound for $g(b)$, showing that it is a bounded function. The fact that g is an MIU map is easily deduced from the fact that $f(x)$ is a multiplicative state for all x (it would fail if $f(x)$ were only a state).

We must now show that



commutes. Taking $x \in X$ and $b \in B$, we see

$$\begin{aligned}\mathrm{MStat}(g)(\eta_X(x))(b) &= (\eta_X(x) \circ g)(b) = \eta_X(x)(g(b)) \\ &= g(b)(x) = f(x)(b),\end{aligned}$$

and hence the unit diagram commutes.

To show the uniqueness of g, suppose there were $h\colon B \to \ell^\infty(X)$ that also made the unit diagram commute. By evaluating $\mathrm{MStat}(h)(\eta_X(x))(b)$ we would obtain $g(b)(x) = h(b)(x)$. Since $g(b)$ and $h(b)$ are elements of $\ell^\infty(X)$ and hence functions, this implies $g(b) = h(b)$ by extensionality, and we can then conclude that $g = h$, as required. We have now shown that ℓ^∞ is a left adjoint to MStat. The other two adjunctions are simply the Stone-Čech compactification of a set and Gelfand duality (which is even an equivalence).

Since the triangle consisting of the forgetful functor **CHaus** $\to$ **Set**, MStat and Spec commutes, the triangle for ℓ^∞,$\mathcal{U}$ and C commutes up to isomorphism, *i.e.* $\ell^\infty \cong C \circ \mathcal{U}$ by uniqueness of adjoints (Proposition 0.4.2). □

When we restrict to the full subcategory **FinSet** $\hookrightarrow$ **Set** of finite sets we obtain a functor $\ell^\infty = \mathbb{C}^{(-)}\colon \mathbf{FinSet} \to \mathbf{FdCC^*Alg}^{\mathrm{op}}$. The next result is then a well-known special case of Gelfand duality (Theorem 1.2.4). We elaborate the proof in some detail because it is important to see where the preservation of multiplication plays a role.

Proposition 1.3.2. *The functor* $\mathbb{C}^{(-)}\colon \mathbf{FinSet} \to \mathbf{FdCC^*Alg}^{\mathrm{op}}$ *is an equivalence of categories.*

Proof. It is easy to see that the functor $\mathbb{C}^{(-)}$ is faithful. The crucial part is to see that it is full. So assume we have two finite sets, seen as natural numbers n, m, and a MIU-homomorphism $h\colon \mathbb{C}^m \to \mathbb{C}^n$. For $j \in m$, let $|j\rangle \in \mathbb{C}^m$ be the standard base vector with 1 at the j-th position and 0 elsewhere. Since this $|j\rangle$ is positive, so is $h(|j\rangle)$, and thus we may write it as $h(|j\rangle) = (r_{1j}, \ldots, r_{nj})$, with $r_{ij} \in \mathbb{R}_{\geq 0}$. Because $|j\rangle \cdot |j\rangle = |j\rangle$, and h preserves multiplication, we get $h(|j\rangle) \cdot h(|j\rangle) = h(|j\rangle)$, and thus $r_{ij}^2 = r_{ij}$. This means $r_{ij} \in \{0, 1\}$, so that h is a (binary) Boolean matrix. But h is also unital, and so:

$$1 \quad = \quad h(1) \quad = \quad h(|1\rangle + \cdots + |m\rangle) \quad = \quad h(|1\rangle) + \cdots + h(|m\rangle). \tag{1.3}$$

For each $i \in n$ there is thus precisely one $j \in m$ with $r_{ij} = 1$ — so that h is a "functional" Boolean matrix. This yields the required function $f\colon n \to m$ with $\mathbb{C}^f = h$.

Corollary 1.2.5 says that the functor $\mathbb{C}^{(-)}\colon \mathbf{FinSet} \to \mathbf{FdCC^*Alg}^{\mathrm{op}}$ is essentially surjective on objects, and thus an equivalence. □

This proof demonstrates that preservation of multiplication, as required for "MIU" maps, is a rather strong condition. We make this more explicit.

Corollary 1.3.3. *For* $n \in \mathbb{N}$ *we have* $\mathrm{MStat}(\mathbb{C}^n) \cong n$.

Proof. By identifying $n \in \mathbb{N}$ with the n-element set $n = \{0, 1, \ldots, n-1\} \in$ **FinSet**, we get by Proposition 1.3.2,

$$\mathrm{MStat}(\mathbb{C}^n) = \mathbf{C^*Alg}(\mathbb{C}^n, \mathbb{C}) \cong \mathbf{FinSet}(1, n) \cong n.$$

□

1.4 Discrete Probabilistic Computations

We turn to probabilistic computations and will see that we remain in the world of commutative C*-algebras, but with PU-maps (positive unital) instead of MIU-maps. Recall that the set of states $\mathrm{Stat}(A)$ of a C*-algebra A contains the PU-maps $A \to \mathbb{C}$.

We summarize here the definition of the expectation monad given in [63]. If $[0,1]^X$ is the effect module of functions from X to $[0,1]$ with pointwise operations, $\mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$. On maps, this is defined as

$$\mathcal{E}(f : X \to Y)(\phi \in \mathcal{E}(X))(b \in [0,1]^Y) = \phi(b \circ f).$$

The unit $\eta_X : X \to \mathcal{E}(X)$ is evaluation, defined as $\eta_X(x)(a) = a(x)$ for $a \in [0,1]^X$. The multiplication $\mu_X : \mathcal{E}^2(X) \to \mathcal{E}(X)$ is defined for $\Phi : [0,1]^{\mathcal{E}(X)} \to [0,1]$, $a \in [0,1]^X$ as

$$\mu_X(\Phi)(a) \quad = \quad \Phi\big(\phi \in \mathcal{E}(X) \mapsto \phi(a)\big). \tag{1.4}$$

This is proven to define a monad in [63, §4].

Lemma 1.4.1. *Sending a set X to the set of states of the C*-algebra $\ell^\infty(X)$ yields the (underlying functor of the) expectation monad $\mathcal{E}$ from [63]: the mapping $X \mapsto \mathrm{Stat}(\ell^\infty(X))$ is isomorphic to the expectation monad $\mathcal{E} : \mathbf{Set} \to \mathbf{Set}$, defined in [63] via effect module homomorphisms: $\mathcal{E}(X) = \mathbf{EMod}\big([0,1]^X, [0,1]\big)$.*

As a result, $\mathrm{Stat}(\mathbb{C}^n) \cong \mathcal{D}(n)$*, for $n \in \mathbb{N}$, where $\mathcal{D}(n)$ is the standard $(n-1)$-simplex.*

Proof. The predicate/effect functor $[0,1]_{(-)} : \mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{EMod}$ is full and faithful by Lemma 1.2.11, and so:

$$\begin{aligned} \mathrm{Stat}(\ell^\infty(X)) \quad = \quad \mathbf{C^*Alg}_{\mathrm{PU}}\big(\ell^\infty(X), \mathbb{C}\big) \quad &\cong \quad \mathbf{EMod}\big([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{C}}\big) \\ &= \quad \mathbf{EMod}\big([0,1]^X, [0,1]\big) \quad = \quad \mathcal{E}(X). \end{aligned}$$

The isomorphism $\alpha : \mathbf{C^*Alg}_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \xrightarrow{\cong} \mathcal{D}(n)$ follows because the expectation and distribution monad coincide on finite sets, see [63]. Explicitly, it is given by $\alpha(\phi)(i \in n) = \phi(|i\rangle)$ and $\alpha^{-1}(\varphi)(a) = \sum_i \varphi(i) \cdot a(i)$. □

In the following, we use θ to refer to the map $\mathbf{BEMod}([0,1]_A, [0,1]_B) \to \mathbf{C^*Alg}_{\mathrm{PU}}(A, B)$ that exists by Lemma 1.2.11.

Proposition 1.4.2. *The expectation monad* $\mathcal{E}(X) \cong \mathbf{C^*Alg}_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C})$ *gives rise to a full and faithful functor:*

$$\begin{array}{rcl} \mathcal{K}\ell(\mathcal{E}) & \xrightarrow{\quad \mathcal{C}_\mathcal{E} \quad} & \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}} \\ X & \longmapsto & \ell^\infty(X) \\ \big(X \xrightarrow{f} \mathcal{E}(Y)\big) & \longmapsto & (a \in \ell^\infty(Y) \mapsto (x \in X \mapsto \theta(f(x))(a))). \end{array} \tag{1.5}$$

Proof. First we need to see that $\mathcal{C}_\mathcal{E}(f)$ is well-defined: the function $\mathcal{C}_\mathcal{E}(f)(a) \colon X \to \mathbb{C}$ must be bounded. We can apply Lemma 1.2.3 to the function $f(x) \in \mathbf{C^*Alg}_{\mathrm{PU}}(\ell^\infty(Y), \mathbb{C})$; it yields $\|\theta(f(x))(a)\| \leq 4\|a\|$. As this holds for each $x \in X$, $|\mathcal{C}_\mathcal{E}(f)(a)(x)| = |\theta(f(x))(a)|$ is bounded (by $4\|a\|$) and therefore $\mathcal{C}_\mathcal{E}(f)(a) \in \ell^\infty(X)$. Next, the map $\mathcal{C}_\mathcal{E}(f)$ is a PU-map of C*-algebras via the pointwise definitions of the relevant constructions.

We check that $\mathcal{C}_\mathcal{E}$ preserves (Kleisli) identities and composition. Identities first. Let $a \in [0,1]_X$:

$$\begin{aligned} \mathcal{C}_\mathcal{E}(\mathrm{id}_X)(a)(x) &= \mathcal{C}_\mathcal{E}(\eta_X)(a)(x) \\ &= \theta(\eta_X(x))(a) \\ &= \eta_X(x)(a) \\ &= a(x) \end{aligned}$$

So $\mathcal{C}_\mathcal{E}(\mathrm{id}_X) = \mathrm{id}_{\ell^\infty(X)}$, because the above holds for all $a \in \ell^\infty(X)$ by Lemma 1.2.11. For composition, with $f : X \to \mathcal{E}(Y)$, $g : Y \to \mathcal{E}(Z)$, $c \in \ell^\infty(Z)$ and $x \in X$:

$$\begin{aligned} \mathcal{C}_\mathcal{E}(g \odot f)(c)(x) &= \theta((g \odot f)(x))(c) \\ &= (g \odot f)(x)(c) \\ &= \mu_Z\left(\mathcal{E}(g)(f(x))\right)(c) \\ &= \mathcal{E}(g)(f(x))\left(\phi \in \mathcal{E}(Z) \mapsto \phi(c)\right) \\ &= f(x)\left((\phi \in \mathcal{E}(Z) \mapsto \phi(c)) \circ g\right) \\ &= f(x)\left(y \in Y \mapsto g(y)(c)\right) \\ &= f(x)\left(\mathcal{C}_\mathcal{E}(g)(c)\right) \\ &= \mathcal{C}_\mathcal{E}(f)\left(\mathcal{C}_\mathcal{E}(g)(c)\right)(x) \\ &= \left(\mathcal{C}_\mathcal{E}(f) \circ \mathcal{C}_\mathcal{E}(g)\right)(c)(x). \end{aligned}$$

By applying Lemma 1.2.11 again, this is so for all $c \in \ell^\infty(Z)$, so $\mathcal{C}_\mathcal{E} g \odot f = \mathcal{C}_\mathcal{E}(f) \circ \mathcal{C}_\mathcal{E}(g)$.

The functor $\mathcal{C}_\mathcal{E}$ is faithful by applying extensionality. To see that $\mathcal{C}_\mathcal{E}$ is full, let $g : \ell^\infty(Y) \to \ell^\infty(X)$ be a PU-map. Define $f : X \to \mathcal{E}(Y)$ as

$$f(x)(b) = [0,1]_g(b)(x),$$

where $x \in X$ and $b \in [0,1]^Y$. We have that $f(x)$ is the restriction of a PU map $\ell^\infty(Y) \to \mathbb{C}$, so is an effect module map $\mathbf{EMod}([0,1]^Y, [0,1])$, and therefore an element of $\mathcal{E}(Y)$, so f is a Kleisli morphism. Now, if we take $b \in [0,1]^Y$, $x \in X$, we have

$$\mathcal{C}_\mathcal{E}(f)(b)(x) = \theta(f(x))(b) = f(x)(b) = [0,1]_g(b)(x) = g(b)(x),$$

so $\mathcal{C}_\mathcal{E}(f) = g$ by Lemma 1.2.11. $\square$

We turn to the finite case, like in the previous section. We do so by considering the Kleisli category $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{E})$ obtained by restricting to objects $n \in \mathbb{N}$. Since the expectation monad $\mathcal{E}$ and the distribution monad $\mathcal{D}$ coincide on finite sets, we have $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{E}) \cong \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$. Maps $n \to m$ in this category are probabilistic transition matrices $n \to \mathcal{D}(m)$. This category has been investigated also in [43]. The following equivalence is known, see *e.g.* [86], although possibly not in this categorical form.

Proposition 1.4.3. *The functor $\mathcal{C}_{\mathcal{E}}$ from* (1.5) *restricts in the finite case to an equivalence of categories:*

$$\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \xrightarrow[\simeq]{\mathcal{C}_{\mathcal{D}}} \mathbf{FdCC^*Alg}_{\mathrm{PU}}^{\mathrm{op}} \tag{1.6}$$

It is given by $\mathcal{C}_{\mathcal{D}}(n) = \mathbb{C}^n$ *and* $\mathcal{C}_{\mathcal{D}}\big(n \xrightarrow{f} \mathcal{D}(m)\big)(a \in \mathbb{C}^m)(i \in n) = \sum_{j \in m} f(i)(j) \cdot v(j)$.

This equivalence (1.6) may be read as: the category $\mathbf{FdCC^*Alg}_{\mathrm{PU}}$ of finite-dimensional commutative C*-algebras, with positive unital maps, is equivalent to the *Lawvere theory* of the distribution monad $\mathcal{D}$.

Proof. Fullness and faithfulness of the functor $\mathcal{C}_{\mathcal{D}}$ follow from Proposition 1.4.2, using the isomorphism $\mathbf{C^*Alg}_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathcal{D}(n)$ from Lemma 1.4.1. This functor $\mathcal{C}_{\mathcal{D}}$ is essentially surjective on objects by Corollary 1.2.5, using the fact that a MIU-map is a PU-map. □

1.5 Continuous Probabilistic Computations

The question arises if the full and faithful functor $\mathcal{K}\ell(\mathcal{E}) \to \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$ from Proposition 1.4.2 can be turned into an equivalence of categories, but not just for the finite case like in Proposition 1.4.3. In order to make this work we have to lift the expectation monad $\mathcal{E}$ on **Set** to the category **CHaus** of compact Hausdorff spaces. For this purpose we use what we call the *Radon* monad $\mathcal{R}$, defined on $X \in \mathbf{CHaus}$ as:

$$\mathcal{R}(X) \quad = \quad \mathrm{Stat}(C(X)) \quad = \quad \mathbf{C^*Alg}_{\mathrm{PU}}\big(C(X), \mathbb{C}\big), \tag{1.7}$$

where, as usual, $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$; notice that the functions $f \in C(X)$ are automatically bounded, since X is compact. We have implicitly applied the forgetful functor from $\mathbf{CCL} \to \mathbf{CHaus}$ to make $\mathcal{R}$ into an endofunctor of **CHaus**. The elements of $\mathcal{R}(X)$ are related to measures in the following way. If μ is a probability measure on the Borel sets of X, integration of continuous functions with respect to μ gives a function $\int_X - d\mu \in \mathcal{R}(X)$. A Radon probability measure, or an inner regular probability measure, is one such that $\mu(S) = \sup_{K \subseteq S} \mu(K)$ where K ranges over compact sets. The map from measures to elements of $\mathcal{R}(X)$ is a bijection [114, Thm. 2.14], and accordingly we shall sometimes refer to elements of $\mathcal{R}(X)$ as measures. Therefore the Radon monad can be considered as a variant of the Giry monad. In fact there are two Giry monads, one on measurable spaces and one on Polish spaces. The Radon monad differs from the measurable space Giry monad in that it uses the topology of a space, and that in the case of a space that is not a standard Borel space there can be non-Radon measures [42, 434K (d), page 192] [49, §53.10, page 231]. The Radon monad differs from the Polish space Giry monad essentially only in the choice of spaces, and on compact Polish spaces they agree, as the topology Giry used is the same as the weak-* topology, and Polish spaces do not admit any non-Radon Borel probability measures[15, Theorems 1.1 and 1.4].

This Radon monad $\mathcal{R}$ is not new: we shall see later that it occurs in [126, Theorem 3] as the monad of an adjunction ("probability measure" is used to mean "Radon probability measure" in that article). It is proven to be a monad independently of that paper in [37, Theorem 2.13], and it has been used more recently in [90]. However, our duality result below — Theorem 1.5.1 — is not known in the literature.

From Proposition 1.2.6 it is immediate that $\mathcal{R}(X)$ is again a compact Hausdorff space. On continuous maps $f : X \to Y$ is defined as

$$\mathcal{R}(f)(\phi)(b) = \phi(b \circ f),$$

where $\phi \in \mathcal{R}(X)$ and $b \in C(Y)$. The unit $\eta_X : X \to \mathcal{R}(X)$ and multiplication $\mu_X : \mathcal{R}^2(X) \to \mathcal{R}(X)$ are defined similarly to the expectation monad, namely as $\eta(x)(a) = a(x)$ and $\mu(\Phi)(a) = \Phi(\psi \mapsto \psi(a))$. We check that η_X is continuous. Recall from the proof of Proposition 1.2.6 that a basic open in $\mathcal{R}(X)$ is of the form $\mathrm{ev}_s^{-1}(U) = \{h \in \mathcal{R}(X) \mid h(s) \in U\}$, where $s \in C(X)$ and $U \subseteq \mathbb{C}$ is open. Then:

$$\eta_X^{-1}\big(\mathrm{ev}_s^{-1}(U)\big) = \{x \in X \mid \eta_X(x)(s) \in U\} = \{x \in X \mid s(x) \in U\} = s^{-1}(U).$$

The latter is an open subset of X since $s\colon X \to \mathbb{C}$ is a continuous function.

We are now ready to state our main, new duality result. It may be understood as a probabilistic version of Gelfand duality, for commutative C*-algebras with PU maps instead of the MIU maps originally used (see Theorem 1.2.4).

Theorem 1.5.1. *The Radon monad* (1.7) *yields an equivalence of categories:*

$$\mathcal{K}\ell(\mathcal{R}) \quad \simeq \quad \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}.$$

Proof. We define a functor $\mathcal{C}_{\mathcal{R}}\colon \mathcal{K}\ell(\mathcal{R}) \to \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}}$ like in (1.5), namely by:

$$\mathcal{C}_{\mathcal{R}}(X) \quad = \quad C(X) \qquad \mathcal{C}_{\mathcal{R}}(f)(b)(x) \quad = \quad f(x)(b).$$

We must first show that $\mathcal{C}_{\mathcal{R}}(f)(b) \in C(X)$. If $(x_i)_{i\in I}$ is a net converging to a point $x \in X$, we want to show $\mathcal{C}_{\mathcal{R}}(f)(b)(x_i) \to \mathcal{C}_{\mathcal{R}}(f)(b)(x)$. We have

$$\mathcal{C}_{\mathcal{R}}(f)(b)(x_i) = f(x_i)(b).$$

As f is continuous, $f(x_i) \to f(x)$, and as evaluating at b is continuous in the weak-* topology, we have

$$f(x_i)(b) \to f(x)(b) = \mathcal{C}_{\mathcal{R}}(f)(b)(x).$$

So $\mathcal{C}_{\mathcal{R}}(f)(b)$ is continuous, and therefore an element of $C(X)$.

As in the ℓ^∞ case, pointwiseness of the operations implies that $\mathcal{C}_{\mathcal{R}}(f)$ is a PU-map. The proof that $\mathcal{C}_{\mathcal{R}}$ is a functor is similar to the proof in Proposition1.4.2. The proof that $\mathcal{C}_{\mathcal{R}}$ is faithful is by functional extensionality and is immediate.

We show that $\mathcal{C}_{\mathcal{R}}$ is full as follows. Let $g : C(Y) \to C(X)$ be a PU-map. Define $f : X \to \mathcal{R}(Y)$ as

$$f(x)(b) = g(b)(x),$$

where $x \in X$ and $b \in C(Y)$. We have that $f(x) \in \mathcal{R}(Y)$ by the pointwiseness of the operations. We show that f is continuous from X to the weak-* topology on $\mathcal{R}(Y)$ as follows. Let $(x_i)_{i\in I}$ be a net converging to $x \in X$. For all $b \in C(Y)$, we have $f(x_i)(b) = g(b)(x_i)$. As $g(b) \in C(X)$, we have $g(b)(x_i) \to g(b)(x) = f(x)(b)$. As this is so for all $b \in C(Y)$, we have $f(x_i) \to f(x)$ in the weak-* topology.

We have shown that f is a Kleisli map, so we only need to show that $\mathcal{C}_{\mathcal{R}}(f) = g$ to show fullness. We have

$$\mathcal{C}_{\mathcal{R}}(f)(b)(x) = f(x)(b) = g(b)(x).$$

The functor is essentially surjective on objects by ordinary Gelfand duality (Theorem 1.2.4), because *-homomorphisms are also PU-maps. □

We investigate the Radon monad $\mathcal{R}$ a bit further, in particular its relation to the distribution monad $\mathcal{D}$ on **Set**.

Lemma 1.5.2. *There is a monad functor* $(U, \tau)\colon \mathcal{D} \to \mathcal{R}$ *in:*

$$\overset{\mathcal{R}\,\curvearrowright}{\mathbf{CHaus}} \xrightarrow{\;U\;} \overset{\curvearrowleft\,\mathcal{D}}{\mathbf{Set}} \qquad\qquad \mathcal{D}U \overset{\tau}{\Longrightarrow} U\mathcal{R}$$

where U is the forgetful functor and τ commutes appropriately with the units and multiplications of the monads $\mathcal{D}$ and $\mathcal{R}$. (Such a map is called a "monad functor" in [125, §1].)

As a result the forgetful functor lifts to the associated categories of Eilenberg-Moore algebras:

$$\begin{array}{ccc} \mathcal{EM}(\mathcal{R}) & \longrightarrow & \mathcal{EM}(\mathcal{D}) \\ \big(\mathcal{R}(X) \overset{\alpha}{\to} X\big) & \longmapsto & \big(\mathcal{D}(UX) \overset{\tau}{\to} U\mathcal{R}(X) \overset{U\alpha}{\to} UX\big) \end{array}$$

Therefore the carrier of an $\mathcal{R}$-algebra is a convex compact Hausdorff space, and every algebra map is an affine function.

Proof. For $X \in$ **CHaus** and $\varphi \in \mathcal{D}(UX)$, that is for $\varphi\colon UX \to [0, 1]$ with finite support and $\sum_{x\in X} \varphi(x) = 1$, we define $\tau_X(\varphi) \in U\mathcal{R}(X)$ on $a \in C(X)$ as:

$$\tau_X(\varphi)(a) \quad = \quad \sum_{x\in X} \varphi(x) \cdot a(x) \;\in\; \mathbb{C}. \tag{1.8}$$

It is easy to see that τ is a linear map $C(X) \to \mathbb{C}$ that preserves positive elements and the unit. Moreover, it commutes appropriately with the units and multiplications. For instance:

$$\big(\tau_X \circ \eta^{\mathcal{D}}_{UX}\big)(x)(a) \quad = \quad \tau_X(\delta_x)(a) \quad = \quad a(x) \quad = \quad U(\eta^{\mathcal{R}}_X)(x)(a).$$

□

The continuous dual space of $C(X)$ can be ordered using (1), by taking the positive cone to be those linear functionals that map positive functions to positive numbers (the dual cone of $C(X)_+$, see Lemma 0.3.8).

Definition 1.5.3. *A state $\phi \in \mathcal{R}(X) = \mathbf{C^*Alg}_{\mathrm{PU}}(C(X), \mathbb{C})$ is a* pure state *if for for each positive linear functional such that $\psi \leq \phi$,* i.e. *such that $\phi - \psi$ is positive, there exists an $\alpha \in [0, 1]$ such that $\psi = \alpha\phi$.* □

Lemma 1.5.4. *For a compact Hausdorff space X, the subset of Dirac measures $\{\eta(x) \mid x \in X\} \subseteq \mathcal{R}(X)$ is exactly the set of pure states and therefore the set of extreme points of the set of Radon measures $\mathcal{R}(X)$ — where $\eta(x) = \eta^{\mathcal{R}}(x)$ is the unit of the monad $\mathcal{R}$.*

Proof. We rely on the basic fact, see [29, 2.5.2, page 43], that a measure is a Dirac measure iff it is a pure state. We prove the above lemma by showing that the pure states are precisely the extreme points of the convex set $\mathcal{R}(X)$.

- If $\phi \in \mathcal{R}(X)$ is a pure state, suppose $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$, a convex combination of two states $\phi_i \in \mathcal{R}(X)$ with $\alpha_i \in [0, 1]$ satisfying $\alpha_1 + \alpha_2 = 1$, where no two elements of $\{\phi, \phi_1, \phi_2\}$ are the same. Then $\phi \geq \alpha_1\phi_1$, since for a positive function $f \in C(X)$ one has $(\phi - \alpha_1\phi_1)(f) = \alpha_2\phi_2(f) \geq 0$. Thus $\alpha_1\phi_1 = \alpha\phi$, for some $\alpha \in [0, 1]$, since ϕ is pure. Then $\alpha_1 = \alpha_1\phi_1(1) = \alpha\phi(1) = \alpha$. If $\alpha_1 = 0$, then $\alpha_2 = 1$ and so $\phi = \phi_2$. If $\alpha_1 > 0$, then $\phi = \phi_1$. Hence ϕ is an extreme point.

- Suppose ϕ is an extreme point of $\mathcal{R}(X)$, *i.e.* that $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$ implies ϕ_1 or $\phi_2 = \phi$. Then if there is a positive linear functional $\psi \leq \phi$, we may take $\alpha_1 = \psi(1) \geq 0$; since $\alpha_1 = \psi(1) \leq \phi(1) = 1$, we get $\alpha_1 \in [0, 1]$. If $\alpha_1 = 0$, then since $\|\psi\| = \psi(1) = 0$ we get $\psi = 0$ and $\psi = 0 \cdot \phi$. If $\alpha_1 = 1$, then $(\phi - \psi)(1) = 0$, which since $\phi - \psi$ was assumed to be positive implies $\phi - \psi = 0$ and hence $\psi = 1 \cdot \phi$. Having dealt with those cases, we have that $\alpha_1 \in (0, 1)$, and so we have a state $\phi_1 = \frac{1}{\alpha_1}\psi$. We may take $\alpha_2 = 1 - \alpha_1 \in (0, 1)$ and obtain a second state $\phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. By construction we have a convex decomposition of $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$. Therefore either $\phi = \phi_1 = \frac{1}{\alpha_1}\psi$ or $\phi = \phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. In the first case, $\psi = \alpha_1\phi$, making ϕ pure. But also in the second case ϕ is pure, since we have $\alpha_2\phi = \phi - \psi$ and thus $\psi = (1 - \alpha_2)\phi$. □

Lemma 1.5.5. *Let X be a compact Hausdorff space.*

(i) The maps $\tau_X\colon \mathcal{D}(UX) \to U\mathcal{R}(X)$ *from* (1.8) *are injective; as a result, the unit/Dirac maps* $\eta\colon X \to \mathcal{R}(X)$ *are also injective.*

(ii) The maps $\tau_X\colon \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$ *embed* $\mathcal{D}(UX)$ *as a dense subset of* $U\mathcal{R}(X)$*.*

Proof. For the first point, assume $\varphi, \psi \in \mathcal{D}(UX)$ satisfying $\tau(\varphi) = \tau(\psi)$. We first show that the finite support sets are equal: $\mathrm{supp}(\varphi) = \mathrm{supp}(\psi)$. Since X is Hausdorff, singletons are closed, and hence finite subsets too. Suppose $\mathrm{supp}(\varphi) \not\subseteq \mathrm{supp}(\psi)$, so that $S = \mathrm{supp}(\varphi) - \mathrm{supp}(\psi)$ is non-empty. Since S and $\mathrm{supp}(\psi)$ are disjoint closed subsets, there is by Urysohn's lemma a continuous function $f\colon X \to [0,1]$ with $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in \mathrm{supp}(\psi)$. But then $\tau(\psi)(f) = 0$, whereas $\tau(\varphi)(f) \neq 0$.

Now that we know $\mathrm{supp}(\varphi) = \mathrm{supp}(\psi)$, assume $\varphi(x) \neq \psi(x)$, for some $x \in \mathrm{supp}(\varphi)$. The closed subsets $\{x\}$ and $\mathrm{supp}(\varphi) - \{x\}$ are disjoint, so there is, again by Urysohn's lemma a continuous function $f\colon X \to [0,1]$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in \mathrm{supp}(\varphi)$. But then $\varphi(x) = \tau(\varphi)(f) = \tau(\psi)(f) = \psi(x)$, contradicting the assumption.

We can conclude that the unit $X \to \mathcal{R}(X)$ is also injective, since its underlying function can be written as the composite $U(\eta^{\mathcal{R}}) = \tau \circ \eta^{\mathcal{D}}\colon UX \rightarrowtail \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$, because τ is a map of monads.

To show that the image of τ_X is dense, we proceed as follows. By Lemmas 1.5.4 and 1.5.2, the extreme points of $\mathcal{R}(X)$ are

$$\{\eta^{\mathcal{R}}(x) \mid x \in X\} \quad = \quad \{\tau\big(\eta^{\mathcal{D}}(x)\big) \mid x \in X\}$$

and are thus in the image of $\tau\colon \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$. Since every convex combination of $\eta^{\mathcal{R}}(x)$ comes from a formal convex sum $\varphi \in \mathcal{D}(UX)$, all convex combinations of extreme points are in the image of τ_X. Using Proposition 1.2.6, $\mathcal{R}(X)$ can be considered an object of **CCL**, i.e. a compact convex subset of a locally convex space. Accordingly, we may apply the Krein-Milman theorem [24, Proposition 7.4, page 142] to conclude the set of convex combinations of extreme points is dense. □

Lemma 1.5.6. *Let* X, Y *be compact Hausdorff spaces. The structure map of each Eilenberg-Moore algebra* $\alpha\colon \mathcal{R}(X) \to X$ *is a* $\mathcal{D}$*-affine function. For each continuous map* $f : X \to Y$*, the function* $\mathcal{R}(f)\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ *is* $\mathcal{D}$*-affine.*

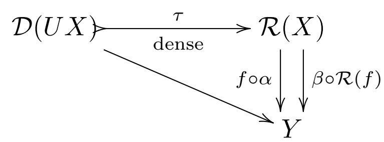
Proof. This follows from naturality of $\tau\colon \mathcal{D}U \Rightarrow U\mathcal{R}$. □

Proposition 1.5.7. *Let* $\alpha\colon \mathcal{R}(X) \to X$ *and* $\beta\colon \mathcal{R}(Y) \to Y$ *be two Eilenberg-Moore algebras of the Radon monad* $\mathcal{R}$*. A function* $f : X \to Y$ *is an algebra homomorphism if and only if* f *is both continuous and affine.*

As a result, the functor $\mathcal{EM}(\mathcal{R}) \to \mathcal{EM}(\mathcal{D})$ *from Lemma 1.5.2 is faithful, and an* $\mathcal{EM}(\mathcal{D})$ *map comes from an* $\mathcal{EM}(\mathcal{R})$ *map if and only if it is continuous.*

We shall follow the convention of writing $\mathrm{CAff}(X, Y)$ for the homset of continuous and $\mathcal{D}$-affine functions $X \to Y$.

Proof. Clearly, each algebra map is both continuous and $\mathcal{D}$-affine. For the converse, if $f : X \to Y$ is continuous, it is a map in the category **CHaus** of compact Hausdorff spaces. Since it is $\mathcal{D}$-affine, both triangles commute in:



Since Y is Hausdorff, there is at most one such map. Therefore f is an algebra map. □

The category $\mathcal{EM}(\mathcal{R})$ of Eilenberg-Moore algebras of the Radon monad may thus be understood as a category of convex compact Hausdorff spaces, with affine continuous maps between them. In Chapter 4 we see how to use a result from [126] to relate this to **CCL**, which is a category of "concrete" convex sets. Using this theorem, it will be shown that "observability" conditions like in [63, top of p. 169] always hold for algebras of $\mathcal{R}$.

In the case of the expectation monad $\mathcal{E}$, it is not necessary to use a forgetful functor to relate it to $\mathcal{D}$ as they are both defined on the same category, **Set**. There is a monad morphism $\sigma : \mathcal{D} \Rightarrow \mathcal{E}$ defined, for $X \in$ **Set**, $\phi \in \mathcal{D}(X)$, and $a \in [0,1]^X$ as

$$\sigma_X(\phi)(a) = \sum_{x \in X} \phi(x) \cdot a(x).$$

The proof of this is given in [63, Lemma 21]. There is also a monad morphism $\tau : \mathcal{U} \Rightarrow \mathcal{E}$ defined as follows, with $\mathcal{F} \in U(X)$ and $a \in [0,1]^X$:

$$\tau_X(\mathcal{F})(a) = \mathrm{ch}(\mathcal{U}(a)(\mathcal{F})),$$

where ch is the unit interval's $\mathcal{EM}(\mathcal{U})$ structure arising from its being a compact Hausdorff space in its usual topology, as described in Example 0.4.11. The proof that this is a monad morphism is detailed in [63, Proposition 16].

For later reference, we summarize these results as follows

Proposition 1.5.8. *There exist monad morphisms $\tau : \mathcal{U} \Rightarrow \mathcal{E}$ and $\sigma : \mathcal{D} \Rightarrow \mathcal{E}$. These induce forgetful functors $\mathcal{EM}(\mathcal{E}) \to \mathcal{EM}(\mathcal{U}) \simeq$ **CHaus**, showing that every $\mathcal{E}$-algebra is canonically a compact Hausdorff space and every map of $\mathcal{E}$-algebras is continuous, and $\mathcal{EM}(\mathcal{E}) \to \mathcal{EM}(\mathcal{D})$, showing that every $\mathcal{E}$-algebra is canonically an abstract convex set, and every map of $\mathcal{E}$-algebras is affine.* □

In Chapter 4 we will see that $\mathcal{EM}(\mathcal{E})$ is in fact equivalent to **CCL**.

Chapter 2

Base-Norm Spaces

2.1 Introduction

In this chapter, we consider the notion of a *base-norm space* and its relationship to convex sets, the distribution monad, and order-unit spaces. Just as order-unit spaces are non-multiplicative, order-theoretic generalizations of the notion of C*-algebra and W*-algebra (Proposition 1.2.10 is a justification of this view), base-norm spaces are similar generalizations of the dual of a C*-algebra, the Banach space containing the state space, and the predual of a W*-algebra, which likewise contains the normal state space (up to isomorphism).

In the literature, there are several definitions of base-norm space, falling into three equivalence classes. Only one of these equivalence classes of definitions is suitable for duality with order-unit spaces, as we shall see, and this is what we choose to call a base-norm space (forming a category **BNS**). The kind of space corresponding to the least strict notion of "base norm space" used in the literature is what we call a pre-base-norm space (forming a category **PreBNS**).

We also show that we can embed any bounded convex set into a pre-base-norm space and this forms an equivalence of categories. We then extend this to show that any sequentially complete bounded convex set embeds as the base of a Banach base-norm space, and this is a restriction of the previous equivalence of categories. By doing so, we relate the base-norm space to generalized probabilistic theories from Davies and Lewis [26] and Edwards [34] to the more recent convex-set-based approach [12, 11], similarly to [133] and [9]. Unfortunately, for reasons of space, we cannot treat tensor products of base-norm spaces and order-unit spaces. If we did they would go along the lines of [95] and [133].

We construct the left adjoints to the functors B : **PreBNS** $\to$ **Set** and B : **BBNS** $\to$ **Set** and show that the monads arising from these adjunctions are the familiar discrete distribution monads, and that the comparison functors to $\mathcal{EM}(\mathcal{D})$ and $\mathcal{EM}(\mathcal{D}_\infty)$ respectively are full and faithful. We then construct the functor BAff, taking bounded affine functions on a $\mathcal{D}$-algebra, and show that when this is applied to the base of a pre-base-norm space it gives the dual space.

Finally, we show that taking the dual space defines a functor $F : \mathbf{PreBNS} \to \mathbf{OUS}^{\mathrm{op}}$ and $G : \mathbf{OUS}^{\mathrm{op}} \to \mathbf{PreBNS}$, and F is a left adjoint to G, in a variant of the adjunctions defined in [56, Theorem 17] and [63, Proposition 5]. We then briefly discuss how this can be restricted to an equivalence. In later chapters this adjunction will be generalized in two different ways to give two equivalences of categories.

2.2 Definitions

2.2.1 Base-Norm Spaces

A *pre-base-norm* consists of a triple (E, E_+, τ), where (E, E_+) is a directed ordered vector space, and $\tau : E \to \mathbb{R}$ is positive linear functional that is not the zero linear functional unless $E = \{0\}$. The map τ is called the trace and is subject to another axiom that we describe below. The *base* is

$$B = \tau^{-1}(1) \cap E_+.$$

The reason for this definition can be seen by considering C^*-algebras. If $E = A^*$, for A a (unital) C^*-algebra, and $\tau(\phi) = \phi(1)$ for $\phi \in A^*$, B is the state space of A. In the commutative case the base is the set of Radon probability measures sitting inside the vector space of signed Radon measures.

Given the base B, we define the *unit ball* U to be the absolutely convex hull of B. In the case that B is non-empty, by Lemma 0.1.1 this can equivalently be defined as

$$U = \mathrm{co}(B \cup -B).$$

For E to be a *pre-base-norm space* we require U to be radially bounded, *i.e.* each ray in E intersects U in a bounded subset (considering the ray as isomorphic to $\mathbb{R}$).

In summary, a pre-base-norm space is a triple (E, E_+, τ) such that (E, E_+) is a directed ordered vector space, τ is a positive linear functional, non-zero if $E \neq \{0\}$, and U is radially bounded.

To live up to their name, pre-base-norm spaces should have an intrinsic notion of norm. We therefore want to show that U is absorbent so that we can define the norm as its Minkowski functional, which will then be a norm by Lemma 0.1.2. We must take a slight detour first.

Lemma 2.2.1. *For any pre-base-norm space, if B is empty, $E = \{0\}$.*

Proof. Suppose for a contradiction that B is empty but $E \neq \{0\}$. Since τ is a trace, we must have $\tau \neq 0$, which means there is $x \in E$, $x \neq 0$, such that $\tau(x) \neq 0$. Since (E, E_+) is directed, $x = x_+ - x_-$ for $x_+, x_- \in E_+$, and at least one of $y = x_\pm$ must be non-zero and satisfy $\tau(y) \neq 0$. But then $z = \frac{y}{\tau(y)}$ is in B because z is positive and

$$\tau(z) = \tau\left(\frac{y}{\tau(y)}\right) = \frac{\tau(y)}{\tau(y)} = 1,$$

contradicting our initial assumption. □

We say a positive linear functional is *strictly* positive if $x \in E_+$, $\tau(x) = 0$ implies $x = 0$.

Lemma 2.2.2. *The trace $\tau : E \to \mathbb{R}$ on any pre-base-norm space is strictly positive.*

Proof. If $E = \{0\}$, then the only possible τ is 0, which is strictly positive because $x \in E$ is always 0. So we therefore consider the case $E \neq \{0\}$. Suppose for a contradiction that τ is not strictly positive. Then there is an $x \in E_+$ such that $\tau(x) = 0$ but $x \neq 0$. By Lemma 2.2.1, there is some $y \in B$.

By the linearity of the trace, $\alpha x + y \in B$ for all $\alpha \in [0, \infty)$, and $\beta x - y \in -B$ for all $\beta \in (-\infty, 0]$ similarly. Therefore

$$U \ni \frac{1}{2}(\alpha x + y) + \frac{1}{2}(\beta x - y) = \left(\frac{1}{2}\alpha + \frac{1}{2}\beta\right) x.$$

If we have $\gamma \in \mathbb{R}$, we can write it as $(\frac{1}{2}(2\gamma) + \frac{1}{2}0)$ if $\gamma \geq 0$ or $(\frac{1}{2}0 + \frac{1}{2}(2\gamma)$ if $\gamma \leq 0$, and so U contains the whole of a non-trivial ray, contradicting radial boundedness of U. □

Lemma 2.2.3. *The set U in a pre-base-norm space is absorbent.*

Proof. Let $x \in E$. We need to find $\alpha \in [0, \infty)$ such that $x \in \alpha U$. Take the decomposition $x = x_+ - x_-$ for $x_+, x_- \in E_+$. Define $\tau(x_+) = \beta$ and $\tau(x_-) = \gamma$. If $\beta \neq 0$, we have $\tau(\frac{x_+}{\beta}) = 1$, so $x_+ \in \beta B$ and hence $x_+ \in \beta U$. If $\beta = 0$, by strict positivity (Lemma 2.2.2) $x_+ = 0$, so we can redefine $\beta = 1$ and hence $x_+ \in \beta U$ in this case as well. Similarly, we have $x_- \in \gamma U$.

Define $\alpha = 2\max\{\beta, \gamma\}$. By absolute convexity of U, we have $x_+ \in \frac{\alpha}{2}U$ and $x_- \in \frac{\alpha}{2}U$, and hence $2x_\pm \in \alpha U$. We can then apply absolute convexity of U again to conclude that

$$x = \frac{1}{2}(2x_+) - \frac{1}{2}(2x_-) \in \alpha U,$$

as required. □

Thus the Minkowski functional $\|\text{-}\|_U$ is always a norm in a pre-base-norm space. We can now define a *base-norm space* – it is a pre-base-norm space in which the positive cone is $\|\text{-}\|_U$-closed. We call a base-norm space a *Banach base-norm space* if it is complete in this norm (it is also sometimes called simply a complete base-norm space). The analogous notion for pre-base-norm spaces is a *Banach pre-base-norm space*, though this is less useful.

We can now define morphisms of (pre-)base-norm spaces. If (E, E_+, τ) and (F, F_+, σ) are (pre-)base-norm spaces, a morphism $f : E \to F$ is a linear, positive map that preserves the trace, *i.e.* $\tau = \sigma \circ f$. Using these morphisms we form the category **PreBNS**, its full subcategory on base-norm spaces **BNS**, and its full subcategory on Banach base-norm spaces, **BBNS**. These morphisms are the *trace-preserving* morphisms. We also have *trace-reducing* or *trace-decreasing* morphisms, which are required to be positive and for which $\sigma(f(x)) \leq \tau(x)$ for all $x \in E_+$. The category of pre-base-norm spaces and trace-reducing maps will be called $\mathbf{PreBNS}_{\leq 1}$, and $\mathbf{BNS}_{\leq 1}$ and $\mathbf{BBNS}_{\leq 1}$ are the corresponding full subcategories.

Lemma 2.2.4. *For $\alpha \geq 0$, if $x \in \alpha U$, $|\tau(x)| \leq \alpha$. Therefore $\|\tau\|$ in the operator norm is ≤ 1, and so τ is norm-continuous.*

Proof. We show that if $x \in U$, $|\tau(x)| \leq 1$ and the statement follows by scaling. The element x is either 0 or is expressible as an element of $\mathrm{co}(B \cup -B)$. In the first case, $\tau(x) = 0$ and so $|\tau(x)| \leq \alpha$. Therefore we concern ourselves with the second case only from now on.

Since $x \in \mathrm{co}(B \cup -B)$ and B is convex, any convex combination used to express x can be reduced to $\beta x_+ + (1-\beta)x_-$ for $x_+ \in B$ and $x_- \in -B$, for some $0 \leq \beta \leq 1$. Then we have

$$\begin{aligned}\tau(x) = \tau(\beta x_+ + (1-\beta)x_-) = \beta\tau(x_+) + (1-\beta)\tau(x_-) = \beta + (1-\beta)(-1)\\ = 2\beta - 1.\end{aligned}$$

From the constraint on β we deduce

$$0 \leq \beta \leq 1 \Leftrightarrow 0 \leq 2\beta \leq 2 \Leftrightarrow -1 \leq 2\beta - 1 \leq 1 \Leftrightarrow |2\beta - 1| \leq 1,$$

which, combined with the previous statement gives $|\tau(x)| \leq 1$. By applying Lemma 0.1.8 with u and $[-1, 1]$ as the absolutely convex sets, we conclude that $\|\tau\| \leq 1$. □

Corollary 2.2.5. *If $x \in E_+$, $\|x\| = \tau(x)$.*

Proof. We have that $x \in \tau(x)B$, and so $x \in \tau(x)U$. This shows that $\|x\| \leq \tau(x)$. If it were the case that $\|x\|_U < \tau(x)$, then

$$\begin{aligned}\neg \inf\{\lambda > 0 | x \in \lambda U\} \geq \tau(x)\} &\Leftrightarrow \neg(\forall \lambda > 0 . x \in \lambda U \Rightarrow \tau(x) \leq \lambda)\\ &\Leftrightarrow \exists \lambda > 0 . x \in \lambda U \wedge \lambda < \tau(x).\end{aligned}$$

Lemma 2.2.4 shows that $x \in \lambda U$ implies $\tau(x) \leq \lambda$, a contradiction, so we have $\|x\|_U \geq \tau(x)$, and so $\|x\| = \tau(x)$. □

We can show the following in the case of a pre-base-norm space with a radially compact ball. The first statement below is an elaboration of a standard fact about radially compact pre-base-norm spaces ([4, Proposition II.1.14] or [6, Proposition 1.26]), but we give the proof here for ease of reference. The second statement can be proved as a consqeuence of duality results between base-norm and order-unit spaces as defined by Alfsen and Shultz [6, Corollary 1.27], but the proof below is elementary.

Proposition 2.2.6. *Let $(E, E_+\tau)$ be a pre-base-norm space such that the base B is nonempty and $U = \mathrm{absco}(B)$ is radially compact.*

(i) *Every $x \in E$ can be expressed as $\alpha x_+ - (1-\alpha)x_-$ where $\alpha \in [0,1]$, $x_+, x_- \in E_+$ and $\|x_+\| = \|x_-\| = \|x\|$. The α is uniquely determined if $x \neq 0$, and is equal to $\frac{1}{2}(\frac{\tau(x)}{\|x\|}+1)$.*

(ii) *E_+ is closed, and therefore (E, E_+, τ) is a base-norm space.*

Proof.

(i) As U is radially compact, we have that $x \in \|x\|U$ (Lemma 0.1.7). Since B is nonempty, $U = \mathrm{co}(-B \cup B)$ (Lemma 0.1.1). Because B is convex, we can therefore express $x = \alpha x_+ - (1-\alpha)x_-$ with $\alpha \in [0,1]$ and $x_+, x_- \in \|x\|B$. By Corollary 2.2.5, $\|x_\pm\| = \tau(x_\pm) = \|x\|$.

Note that we have

$$\begin{aligned}\tau(\alpha x_+ - (1-\alpha)x_-) &= \alpha\tau(x_+) - (1-\alpha)\tau(x_-)\\ &= \alpha\|x_+\| - (1-\alpha)\|x_-\|\\ &= (2\alpha-1)\|x\|.\end{aligned}$$

If $x \neq 0$, we have $\|x\| \neq 0$ so we can rearrange this expression to get $\alpha = \frac{1}{2}(\frac{\tau(x)}{\|x\|}+1)$. The expression on the right depends only on x, so α is uniquely defined.

(ii) Let (x_i) be a sequence in E_+ converging in the base-norm to $x \in E$. If $x = 0$ then $x \in E_+$, so we reduce to the case that $x \neq 0$. Observe that if this is so $\alpha = 1$ iff $x \in E_+$, because if $\alpha = 1$ then we have $x = x_+ \in E_+$, and if $x \in E_+$ it is expressible as $1 \cdot x - 0 \cdot x$, which by uniqueness of α gives $\alpha = 1$. We also observe that $\|\text{-}\|$ is continuous on E, so $\|\text{-}\|^{-1}$ is continuous on $E \setminus \{0\}$. As τ is continuous (Lemma 2.2.4), we have that $x \mapsto \frac{1}{2}(\frac{\tau(x)}{\|x\|}+1)$ is continuous on $E \setminus \{0\}$. As $x \neq 0$, we can replace (x_i) with a subsequence (y_i) such that $\|x - x_i\| \leq \frac{\|x\|}{2}$ and therefore $y_i \neq 0$ for all i. Then we have $\frac{1}{2}(\frac{\tau(x)}{\|x\|}+1) = 1$ for all y_i and so we must have $\frac{1}{2}(\frac{\tau(x)}{\|x\|}+1) = 1$ for x by continuity, implying $x \in E_+$. □

We now move on to proving facts about morphisms.

Lemma 2.2.7. *A trace-preserving morphism $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ of pre-base-norm spaces maps the base into the base,* i.e. *if B_E is the base of E and B_F the base of F, $f(B_E) \subseteq B_F$.*

Proof. Suppose $x \in B_E$, which is to say that $x \in E_+$ and $\tau(x) = 1$. Since f is positive, $f(x) \in F_+$. By preservation of the trace, we have $\sigma(f(x)) = \tau(x) = 1$. Therefore $f(x) \in B_F$. □

The *sub-base* of a pre-base-norm space (E, E_+, τ) is the set $B^{\leq 1} = E_+ \cap \tau^{-1}((-\infty, 1])$.

Lemma 2.2.8. *We have three equivalent ways to express the sub-base:*

$$E_+ \cap \tau^{-1}((-\infty, 1]) = E_+ \cap \tau^{-1}([0, 1]) = \mathrm{co}(\{0\} \cup B)$$

Proof.

- $E_+ \cap \tau^{-1}((-\infty, 1]) = E_+ \cap \tau^{-1}([0, 1])$:
  We have $E_+ \cap \tau^{-1}([0, 1]) \subseteq E_+ \cap \tau^{-1}((-\infty, 1])$ immediately. By the positivity of τ, if $x \in E_+$, $\tau(x) \geq 0$, so the opposite inclusion also holds.

- $E_+ \cap \tau^{-1}([0, 1]) \subseteq \mathrm{co}(\{0\} \cup B)$:
  Let $x \in E_+ \cap \tau^{-1}([0, 1])$. If $x = 0$, $x \in \mathrm{co}(\{0\} \cup B)$. Contrariwise, if $x \neq 0$, the strict positivity of τ (Lemma 2.2.2) implies $\tau(x) \neq 0$, so we can take $\frac{x}{\tau(x)} \in B$. We may then express x in a manner clearly showing it is a convex combination in $\mathrm{co}(\{0\} \cup B)$:

$$x = \tau(x)\frac{x}{\tau(x)} + (1 - \tau(x))0.$$

- $\mathrm{co}(\{0\} \cup B) \subseteq E_+ \cap \tau^{-1}([0, 1])$:
  Since $x \in \mathrm{co}(\{0\} \cup B)$, and $\{0\}$ and B are both convex, x can be expressed as a convex combination

$$x = \alpha x' + (1 - \alpha)0,$$

  and therefore that $x = \alpha x'$ for $x' \in B$. This implies that $x \in E_+$. Now

$$\tau(x) = \tau(\alpha x') = \alpha\tau(x') = \alpha.$$

  Since $\alpha \in [0, 1]$, this finishes the proof. □

Corollary 2.2.9. *For any pre-base-norm space (E, E_+, τ) and $\alpha \in \mathbb{R}_{>0}$, we have $E_+ \cap \alpha U = \alpha\mathrm{co}(\{0\} \cup B)$.*

Proof. We have

$$\begin{aligned} E_+ \cap \mathrm{Ball}(\|\text{-}\|_U) &= E_+ \cap \tau^{-1}([0, 1]) && \text{Corollary 2.2.5} \\ &= \mathrm{co}(\{0\} \cup B) && \text{Lemma 2.2.8.} \end{aligned}$$

But this is not quite what we need. It is enough to show that

$$E_+ \cap U \subseteq E_+ \cap \mathrm{Ball}(\|\text{-}\|_U) = \mathrm{co}(\{0\} \cup B),$$

using Lemma 0.1.6 for the first inclusion. Then, since $E_+ \cap U$ is a convex set containing B and 0, we have the other inclusion and so $E_+ \cap U = \mathrm{co}(\{0\} \cup B)$.

If $\alpha = 0$, we have $E_+ \cap \alpha U = \{0\} = \alpha\mathrm{co}(\{0\} \cup B)$. If, on the other hand, $\alpha \neq 0$, multiplying by α is a bijection, so

$$\alpha\mathrm{co}(\{0\} \cup B) = \alpha(E_+ \cap U) = \alpha E_+ \cap \alpha U = E_+ \cap \alpha U.$$

□

Lemma 2.2.10. *A trace-reducing map (and hence also a trace-preserving map) preserves the sub-base,* i.e. $f(B_E^{\leq 1}) \subseteq B_F^{\leq 1}$.

Proof. Let $f : (X, X_{,+}, \sigma) \to (Y, Y_+, \tau)$ be a trace-reducing map. Let $x \in B_X^{\leq 1}$, *i.e.* $x \in X_+ \cap \tau^{-1}((-\infty, 1])$. Since f is positive, $f(x) \in Y_+$. Since f is trace-reducing, $\sigma(f(x)) \leq \tau(x) = 1$, so $f(x) \in \sigma^{-1}((-\infty, 1])$ as well. □

Lemma 2.2.11. $\text{absco}(B) = \text{co}(-B^{\leq 1} \cup B^{\leq 1})$

Proof.

- $\text{absco}(B) \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$:

  Consider $x \in \text{absco}(B)$, expressed as an absolutely convex combination:

  $$x = \alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n$$

  with $x_i \in B$, and the indexing chosen so that $\alpha_1, \ldots, \alpha_k \geq 0$ and $\alpha_{k+1}, \ldots, \alpha_n \leq 0$, with both sets of coefficients possibly empty (indicating the empty absolutely convex combination). Define

  $$\beta_+ = \sum_{i=1}^{k} \alpha_i \qquad\qquad \beta_- = \sum_{i=k+1}^{n} -\alpha_i.$$

  These numbers are non-negative and $\beta_+ + \beta_- \leq 1$.

  There are four possible cases, as each β can either be zero or nonzero. If $\beta_+ = \beta_- = 0$, then $x = 0$, so $x \in B^{\leq 1} \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$. If one of them is nonzero, let $s \in \{+, -\}$ be its sign. We have that

  $$x = (1 - \beta_s)0 + \sum_i s\alpha_i x_i$$

  is a convex combination, and so shows that $x \in sB^{\leq 1} \subseteq \text{co}(-B^{\leq 1} \cup B^{\leq 1})$.

  Now suppose that $\beta_+, \beta_- \neq 0$. Define

  $$x_+ = \sum_{i=1}^{k} \frac{\alpha_i}{\beta_+} x_i \qquad\qquad x_- = \sum_{i=k+1}^{n} \frac{-\alpha_i}{\beta_-} - x_i.$$

  These are convex combinations, so $x_+ \in B \subseteq B^{\leq 1}$ and $x_- \in -B \subseteq -B^{\leq 1}$. Let $\beta_0 = 1 - \beta_+ - \beta_-$. Define

  $$x'_+ = \frac{\beta_0}{\beta_0 + \beta_+} 0 + \frac{\beta_+}{\beta_0 + \beta_+} x_+ \qquad\qquad \beta'_+ = \beta_0 + \beta_+.$$

  Now, by Lemma 2.2.8 $x'_+ \in B^{\leq 1}$, and we have arranged it so that $\beta'_+ + \beta_- = 1$ and they are both positive. Therefore $\beta'_+ x'_+ + \beta_- x_- \in \text{co}(-B^{\leq 1} \cup$

$B^{\leq 1}$) by definition. We have arranged it so that

$$\begin{aligned}\beta_+ x_+ + \beta_- x_- &= (\beta_0 + \beta_+)\left(\frac{\beta_0}{\beta_0+\beta_+}0 + \frac{\beta_+}{\beta_0+\beta_+}x_+\right) + \beta_- x_- \\ &= \beta_+ x_+ + \beta_- x_- \\ &= \beta_+\left(\sum_{i=1}^{k}\frac{\alpha_i}{\beta_+}x_i\right) + \beta_-\left(\sum_{i=k+1}^{n}\frac{-\alpha_i}{\beta_-}-x_i\right) \\ &= \sum_{i=1}^{k}\alpha_i x_i + \sum_{i=k+1}^{n}\alpha_i x_i = x.\end{aligned}$$

- $\mathrm{co}(-B^{\leq 1} \cup B^{\leq 1}) \subseteq \mathrm{absco}(B)$:

  We have that $B \subseteq \mathrm{absco}(B)$ and $0 \subseteq \mathrm{absco}(B)$. Since convex combinations are a special case of absolutely convex combinations, Lemma 2.2.8 implies $B^{\leq 1} \subseteq \mathrm{absco}(B)$. Since $-1 \cdot x$ is an absolutely convex combination of x, we have that $-B^{\leq 1} \subseteq \mathrm{absco}(B)$ too. Reapplying the fact that convex combinations are a special case of absolutely convex combinations, we have that $\mathrm{co}(-B^{\leq 1} \cup B^{\leq 1}) \subseteq \mathrm{absco}(B)$. □

Note that the above identity holds even in the case that $E = 0$.

Proposition 2.2.12. *A trace-reducing morphism $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ of pre-base-norm spaces is bounded with operator norm $\|f\| \leq 1$. If f is trace-preserving and $E \neq 0$, $\|f\| = 1$.*

Proof. Let f be trace-reducing. By applying Lemma 2.2.10, we have that $f(B_E^{\leq 1}) \subseteq B_F^{\leq 1}$. By Lemma 2.2.11, we have that $\mathrm{absco}(B_E) = \mathrm{co}(-B_E^{\leq 1} \cup B_E^{\leq 1})$ and likewise for F. By linearity of f, we have that $f(\mathrm{co}(-B_E^{\leq 1} \cup B_E^{\leq 1})) \subseteq \mathrm{co}(-B_F^{\leq 1} \cup B_F^{\leq 1})$. The hypotheses of Lemma 0.1.8 are then satisfied, so we can conclude $\|f\| \leq 1$.

Now, if f is trace-preserving and $E \neq 0$, then by Lemma 2.2.1, $B_E \neq \emptyset$. So if $x \in B_E$, we have that x is positive and of trace 1, so by Corollary 2.2.5 we have $\|x\| = 1$. If $x \in B_E$, by Lemma 2.2.7 $f(x) \in B_F$, and so $\|f(x)\| = 1$ too. Therefore $\|f\|$, since it is an upper bound for $\|f(x)\|$ as x varies over the closed unit ball of E, is greater than or equal to 1. Since $\|f\| \leq 1$ in general, this shows that $\|f\| = 1$. □

We therefore have that f is continuous, and that there exist forgetful functors U_1 : **PreBNS** $\to$ **Normed**$_1$ and U_∞ : **PreBNS** $\to$ **Normed**, where **Normed**$_1$ is the *metric* category of normed spaces, having maps of operator norm ≤ 1 (called *contractions*) as maps, and **Normed** is the topological category of normed spaces, with bounded maps. These functors restrict to functors U_1 : **BBNS** $\to$ **Ban**$_1$ and U_∞ : **BBNS** $\to$ **Ban**, where the **Ban**$_1$ and **Ban** are full subcategories on Banach spaces.

2.2.2 Bounded Convex Sets

We define a category **BConv** as follows. Its objects are pairs (E, X), where E is a locally convex space and $X \subseteq E$ is a subset that is bounded but also convex. The hom set is defined as

$$\mathbf{BConv}((E, X), (F, Y)) = \{f : X \to Y \mid f \text{ is affine}\}.$$

Note that we do not require the morphism to do anything with the ambient vector spaces, and we do not require any continuity for maps, the topology serves only to define boundedness. The purpose of this category is to package up a standard construction of a pre-base-norm space and morphisms between pre-base-norm spaces constructed in this manner.

Proposition 2.2.13. *Let (E, X) be an object of* **BConv**. *There exists a pre-base-norm space (F, F_+, τ), with a locally convex topology $\mathcal{S}$ in which τ is continuous, such that the topology and uniformity defined by the norm are finer than $\mathcal{S}$ and its uniformity, and a* **BConv** *isomorphism $i : (E, X) \to (F, B_F)$ that is a homeomorphism for the subspace topologies and a uniform isomorphism for the subspace uniformities on X and B_F.*

Proof. If $X = \emptyset$, let F be the zero base-norm space and take $i = \mathrm{id}_\emptyset$. As there is only one topology and uniformity on the empty set, this is a uniform homeomorphism.

We now reduce to the case that $X \neq \emptyset$. Pick a point $b \in X$. We have that $0 \in X - b$, and we take $E' = \operatorname{span}(X - b)$, giving it the subspace topology from E, which is locally convex. We then take $F = \mathbb{R} \times E'$, defining $\mathcal{S}$ to be the locally convex product topology. We define

$$F_+ = \{\alpha(1, x) \mid \alpha \in \mathbb{R}_{\geq 0} \text{ and } x \in X - b\}$$

and $\tau = \pi_1$, which is therefore continuous. We must show that (F, F_+, τ) is a pre-base-norm space. We first show that F_+ is a cone generating F.

- F_+ closed under multiplication by $\alpha \in \mathbb{R}_{\geq 0}$:

  This is immediately apparent from the definition above.

- F_+ is closed under addition:

  Let $\alpha(1, x)$ and $\beta(1, y)$ be elements of F_+. Then

$$\begin{aligned}\alpha(1, x) + \beta(1, y) &= (\alpha + \beta, \alpha x + \beta y)\\ &= (\alpha + \beta)\left(1, \frac{\alpha}{\alpha + \beta}x + \frac{\beta}{\alpha + \beta}y\right),\end{aligned}$$

  and we see that $\alpha + \beta \in \mathbb{R}_{\geq 0}$, and $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in X - b$ because $X - b$ is a convex subset of E'.

- $F_+ \cap -F_+ = \{0\}$:

  Let $\alpha(1,x) = -\alpha(1,x)$. Then in particular, $\alpha = -\alpha$ so $\alpha = 0$ and $\alpha(1,x) = (0,0)$.

- $\operatorname{span}(F_+) = F$:

  Let $(\alpha, x) \in F$. Since $E' = \operatorname{span}(X - x)$, we have that $x = \sum_{i=1}^{n} \alpha_i x_i$ for $\alpha_i \in \mathbb{R}$ and $x_i \in X - x$. We define $x_{n+1} = 0$, as $0 \in X - x$, and $\alpha_{n+1} = \alpha - \sum_{i=1}^{n} \alpha_i$. Then

  $$\sum_{i=1}^{n+1} \alpha_i = \alpha, \text{ and } \sum_{i=1}^{n+1} \alpha_i x_i = x + 0 = x,$$

  so

  $$\sum_{i=1}^{n+1} \alpha_i (1, x_i) = (\alpha, x)$$

  and we have expressed it as a linear combination of elements of F_+.

We then need to show that τ is nonzero and that $\operatorname{absco}(B_F)$ is radially bounded. Since $0 \in X - b$, we have $(1,0) \in F_+$, and $\tau(1,0) = 1$, so $\tau \neq 0$.

Now the base is

$$B_F = \{\alpha(1,x) | x \in X - b\} \cap \tau^{-1}(1),$$

and $\tau(\alpha(1,x)) = 1$ implies that $\alpha = 1$, so

$$B_F = \{(1,x) | x \in X - b\}.$$

i.e. $B_F = \{1\} \times (X - b)$. By Lemma 0.1.13 $X - b$ is a bounded subset of E' and B_F is therefore a bounded subset of F. By Lemma 0.1.15 $\operatorname{absco}(B_F)$ is bounded, and therefore radially bounded (Lemma 0.1.16). This shows that (F, F_+, τ) is a pre-base-norm space.

To show that the pre-base-norm topology is finer than $\mathcal{S}$, let U be a 0-neighbourhood in $\mathcal{S}$. As $\operatorname{absco}(B_F)$ is bounded, there is an $\alpha > 0$ such that $\operatorname{absco}(B_F) \subseteq \alpha U$. By Lemma 0.1.6, the unit ball of F in its pre-base-norm topology is a subset of $2\operatorname{absco}(B_F)$, so $2\operatorname{absco}(B_F)$ is a neighbourhood of zero. Since multiplication by a scalar is a homeomorphism, we have that $2\alpha^{-1}\operatorname{absco}(B_F)$ is a neighbourhood of zero, and therefore U is a neighbourhod of zero. Therefore every $\mathcal{S}$-open set is open in the pre-base-norm topology. Since the basic entourages for the uniformity are defined by $\{(x,y) | x - y \in U\}$ for U a neighbourhood of 0, we have that the uniformity defined by the pre-base-norm is finer than the $\mathcal{S}$-uniformity.

We define $i : X \to B_F$ as $i(x) = (1, x - b)$. We see that if $x \in X$, then $(1, x - b) \in \{1\} \times (X - b) = B_F$, so i has the right type. We can decompose i as $(- + (1,0)) \circ \kappa_2 \circ (- + (-x))$. The first and last part are affine uniform isomorphisms by Lemma 0.1.18, and the middle part is a linear homeomorphism, hence a uniform isomorphism, when restricted to $E' \to \{0\} \times E'$ (Lemma 0.1.17).

Therefore, when restricted to $X \to B_F$, it is an affine uniform isomorphism (and therefore a homeomorphism as well). It is also an isomorphism $(E, X) \to (F, B_F)$ in **BConv**. $\square$

Since the pre-base-norm topology is always finer than the original topology of a bounded convex set, it is the analogous notion for convex sets of the discrete topology on sets.

Given a pre-base-norm space (E, E_+, τ), we have seen that we can define an element of **BConv** as (E, B_E), taking the locally convex topology to be that defined by the norm. Lemma 2.2.7 implies that if we have a trace-preserving morphism $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ then $f|_{B_E}$ restricts to have codomain B_F, and so $f|_{B_E}$ is therefore a map $(E, B_E) \to (F, B_F)$ in **BConv**. This defines a functor $B : \mathbf{PreBNS} \to \mathbf{BConv}$, which is essentially surjective by Proposition 2.2.13, and is faithful by definition.

Lemma 2.2.14. *Let (E, E_+, τ) be a pre-base-norm space, $\mathcal{T}$ a locally convex topology on E such that τ is sequentially continuous. Then B_E is sequentially closed iff E_+ is sequentially closed.*

Proof. If E_+ is sequentially closed, then as $\tau^{-1}(1)$ is sequentially closed, we have $B_E = E_+ \cap \tau^{-1}(1)$ is sequentially closed.

For the other direction, suppose that B_E is sequentially closed, and let (x_i) be a sequence in E_+ converging in $\mathcal{T}$ to $x \in E$. To show E_+ is sequentially closed we must show that x is in E_+. As τ is sequentially continuous, $\tau(x_i) \to \tau(x)$. If $\tau(x) = 0$, then $x = 0$ by Lemma 2.2.2, so $x_i \to 0$, which is an element of E_+. We can therefore reduce to the case that $\tau(x) > 0$. Define (y_i) to be the subsequence of (x_i) starting at the n such that for all $i \geq n$ $|\tau(x_i) - \tau(x)| < \frac{\tau(x)}{2}$, a value that must exist by the convergence of $(\tau(x_i))_i$. We therefore have $y_i \to x$ and $\tau(y_i) > 0$ for all $i \in \mathbb{N}$. Define $z_i = \frac{y_i}{\tau(y_i)}$ and $z = \frac{x}{\tau(x)}$. Since $\text{-}^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is continuous, we have $\frac{1}{\tau(y_i)} \to \frac{1}{\tau(x)}$. By joint continuity of scalar multiplication, we have $\frac{y_i}{\tau(y_i)} \to \frac{x}{\tau(x)}$, *i.e.* $z_i \to z$. Because z_i is a sequence in B_E, we have $z \in B_E$, and therefore $x = \tau(x)z \in E_+$. $\square$

The following lemma is based on [118, V.3.4 Lemma 2] and is stated in this way because it will be used later in another proof.

Lemma 2.2.15. *Let $(E, \|\text{-}\|)$ be a normed space, $U = \mathrm{Ball}(\|\text{-}\|)$, and let $E_+ \subseteq E$ be a cone such that $E_+ \cap U$ is σ-convex. Define $F = E_+ - E_+$ and $V_1 = \mathrm{co}(E_+ \cap U \cup -E_+ \cap U)$, $V_2 = E_+ \cap U - E_+ \cap U$. Then V_1 and V_2 define equivalent norms $\|\text{-}\|_{V_1}, \|\text{-}\|_{V_2}$ on F in which it is complete.*

Proof. We first show that V_1 and V_2 define equivalent norms. We can see that $V_1 \subseteq V_2$ as follows. If $\alpha x_+ - (1-\alpha)x_- \in V_1$, *i.e.* $x_+, x_- \in U \cap E_+$ and $\alpha \in [0, 1]$, then αx_+ and $(1-\alpha)x_-$ are elements of $U \cap E_+$ by absolute convexity of U and E_+ being a cone. Therefore $\alpha x_+ - (1-\alpha)x_- \in E_+ \cap U - E_+ \cap U = V_2$.

We then show that $V_2 \subseteq 2V_1$ as follows. If $x_+ - x_- \in U_2$, *i.e.* $x_+, x_- \in E_+ \cap U$, then $x_+ - x_- = 2(\frac{1}{2}x_+ - \frac{1}{2}x_-) \in 2\mathrm{co}(E_+ \cap U \cup -E_+ \cap U)$.

Both V_1 and V_2 are clearly balanced as their definitions equivalent when negated. Then V_1 is convex by its definition as a convex hull, while for V_2, if we have $x_+ - x_-, y_+ - y_- \in V_2$, and $\alpha \in [0,1]$, then

$$\alpha(x_+ - x_-) + (1-\alpha)(y_+ - y_-) = (\alpha x_+ + (1-\alpha)y_+) - (\alpha x_- + (1-\alpha)y_-) \in V_2$$

by the convexity of $E_+ \cap U$. We also have that $0 \in V_1$ and $0 \in V_2$, so neither of the sets is empty, so V_1 and V_2 are absolutely convex by Lemma A.3.1.

The containment results between $V_1 \subseteq V_2$ and $V_2 \subseteq 2V_2$ show that each is absorbent iff the other is, so we show that V_2 is absorbent (in F). Let $x_+ - x_- \in E_+ - E_+ = F$. As U is absorbent, being the unit ball of a norm, there exist $\alpha, \beta \in \mathbb{R}_{>0}$ such that $x_+ \in \alpha U$ and $x_- \in \beta U$, and these also hold for any greater real number in either case. Therefore, if we take $\gamma = \max\{\alpha, \beta\}$, we conclude that $x_+, x_- \in \gamma U$, so $x_+ - x_- \in E_+ \cap \gamma U - E_+ \cap \gamma U = \gamma V_2$.

This proves that $\|\text{-}\|_{V_1}$ and $\|\text{-}\|_{V_2}$ are norms, and are equivalent. Therefore F is complete in one iff it is complete in the other, so we show that F is complete in $\|\text{-}\|_{V_2}$. Let $(a_i)_{i\in\mathbb{N}}$ be a $\|\text{-}\|_{V_2}$-Cauchy sequence in F. We can select a subsequence[1] (b_i) such that $b_{i+1} - b_i \in 2^{-i}V_2$ for all $i \in \mathbb{N}$. Therefore there exist $x_i, y_i \in E_+ \cap 2^{-i}U$ such that $b_{i+1} - b_i = x_i - y_i$.

We can define $x_i' = 2^i x_i$ and $y_i' = 2^i y_i$, and these are sequences in $E_+ \cap U$. Then $\sum_{i=1}^n x_i = \sum_{i=1}^n 2^{-i}x_i'$, so by the σ-convexity of $E_+ \cap U$, $\sum_{i=1}^\infty x_i$ converges, as does $\sum_{i=1}^\infty y_i$. We call these sums x and y respectively.

Then we can define $b = x - y + b_1 \in F$. To finish the proof that F is complete, we will show that (b_i), and therefore (a_i), converges to b. Given $\epsilon > 0$, there exist j and k such that $\left\|x - \sum_{i=1}^j x_i\right\|_{V_2} < \frac{\epsilon}{2}$ and $\left\|y - \sum_{i=1}^k y_i\right\|_{V_2} < \frac{\epsilon}{2}$. If we take $m = \max j, k$, then for all $n \geq m$ we have that

$$\left\|\left(x - \sum_{i=1}^n x_i\right) + \left(y - \sum_{i=1}^n y_i\right)\right\|_{V_2} < \epsilon, \text{ so } \left\|(x-y) - \sum_{i=1}^n x_i - y_i\right\|_{V_2} < \epsilon.$$

Then

$$\sum_{i=1}^n x_i - y_i = \sum_{i=1}^n b_{i+1} - b_i = b_{n+1} - b_1,$$

as it is a telescoping sum. We can therefore conclude that

$$\|(x - y + b_1) - (b_{n+1} - b_1 + b_1)\|_{V_2} < \epsilon,$$

so $\|b - b_{n+1}\|_{V_2} < \epsilon$. Therefore for all $n \geq m+1$ we have $\|b - b_n\|_{V_2} < \epsilon$, and so (b_i) converges to b in $\|\text{-}\|_{V_2}$. □

[1] By taking b_i to be the a_N where N is the smallest number such that for all $j, k \geq N$ $\|a_j - a_k\|_{V_2} < 2^{-i}$, which necessarily exists for a Cauchy sequence.

2.2.3 Comparison of Definitions

There are other definitions of base-norm space available in the literature. Some are superficial, such as using a hyperplane to define the base of the cone instead of the linear functional τ, as in [4, §II.1 p. 77] [6, p.9]. There is also the definition used by Nagel [94, §2], which is the same as ours except requiring that τ be *strictly* positive and relaxing radially compact to radially bounded. In Asimow and Ellis's definition [8, Definition, p.36] a base-norm space is defined to be a particular kind of normed ordered vector space (with closed positive cone).

In the following, we will give a proof that the Alfsen-Shultz definition coincides with radially compact, non-zero base-norm spaces, a proof that Nagel's definition coincides with pre-base-norm spaces, and two counterexamples – a pre-base-norm space that is not a base-norm space, and a base-norm space that is not radially compact. Asimow and Ellis's definition agrees with ours, except the zero base-norm space, but we leave this as an exercise to the reader.

Nagel's Definition

Nagel's definition [94, §2] is that a base-norm space is a triple (E, E_+, τ), where (E, E_+) is a directed ordered vector space, τ is a strictly positive linear functional $\tau : E \to \mathbb{R}$, and with B having its usual definition, $U = \mathrm{co}(-B \cup B)$ is radially bounded. Because of the use of $U = \mathrm{co}(-B \cup B)$ rather than $U = \mathrm{absco}(B)$, the zero base-norm space does not satisfy Nagel's definition. However, by Lemma 2.2.2 every non-zero pre-base-norm space is a base-norm space in Nagel's sense, and every base-norm space in Nagel's sense is a pre-base-norm space.

However, it is not the case that every pre-base-norm space is a base-norm space. We construct a counter-example as follows, which we call the *strict plane*. Take the underlying vector space to be $E = \mathbb{R}^2$. We take the positive cone E_+ to be

$$E_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\}.$$

This is a cone, as it can be seen to satisfy the axioms by elementary manipulation of inequalities. We show that E_+ is generating as follows. Let $(x, y) \in \mathbb{R}^2$. Each real number z can be expressed as the difference of two strictly positive numbers as follows. Pick some $\epsilon > 0$. If $z > 0$, $z_+ = z+\epsilon$ and $z_- = \epsilon$. If $z = 0$, take $z_+ = \epsilon$ and $z_- = \epsilon$. If $z < 0$, take $z_+ = \epsilon$ and $z_- = -z + \epsilon$. Apply this decomposition independently to x and y, and we have $(x, y) = (x_+, y_+) - (x_-, y_-)$. Since their components are strictly positive, we have that (x_+, y_+) and $(x_-, y_-) \in E_+$, as required. So (E, E_+) is a directed partially ordered vector space.

We define the trace $\tau : E \to \mathbb{R}$ as $\tau(x, y) = x + y$. We can see this is positive and not zero.

We therefore only need to show that $U = \mathrm{co}(B \cap -B)$ is radially bounded.

Lemma 2.2.16. *U is radially bounded.*

Proof. We show this by showing that B is contained in the closed unit ball D for the Hilbert space norm on $\mathbb{R}^2$, which is the unit sphere. Since $U = \mathrm{absco}(B)$

and D is absolutely convex, we can conclude $U \subseteq D$. Then radial boundedness follows from the radial boundedness of D, which comes from the fact that D contains no line through the origin (using Lemma 0.1.2).

We have that

$$\begin{aligned} B &= E_+ \cap \tau^{-1}(1) \\ &= (\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \cup \{(0, 0)\}) \cap \{(x, y) \in \mathbb{R}^2 \mid x + y = 1\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, x + y = 1\}. \end{aligned}$$

We need to show that $x > 0$, $y > 0$, and $x + y = 1$ implies $x^2 + y^2 \leq 1$. We have that $1 = 1^2 = (x + y)^2 = x^2 + 2xy + y^2$. We also have that $2xy > 0$, using the two inequalities. Therefore

$$x^2 + y^2 = 1 - 2xy < 1$$

as required. By the previous paragraph, this is enough to prove the lemma. □

We have now proved that (E, E_+, τ) is a pre-base-norm space, or equivalently a base-norm space in Nagel's sense.

Counterexample 2.2.17. *(E, E_+, τ) is not a base-norm space.*

Proof. By [118, Theorem I.3.2] the norm topology on E agrees with the usual topology on $\mathbb{R}^2$ as it is Hausdorff and finite-dimensional. Therefore we can see that E_+ is not closed in E, so (E, E_+, τ) is not a base-norm space. □

At first, this might seem like a deficiency of the definition we use. But a space with an unclosed positive cone can never occur as a dual cone because dual cones are always closed (Lemma 0.3.7 and Theorem 0.3.9). This makes it reasonable to restrict to those spaces with closed positive cones. Pre-base-norm spaces still have their advantages in certain situations, however, such as when one wants to find a space having a given convex set as its base, as we shall see later.

The Alfsen-Shultz Definition

This definition can be found in [4, §II.1 p. 77] [6, p.9]. We repeat it here. It depends on the definition of the base of a cone, which is found in [6, p. 3][4, p. 76], which we define first. Given a cone $E_+ \subseteq E$, a base $B \subseteq E_+$ is the convex set given by the intersection of a hyperplane H ($0 \notin H$) with E_+, *i.e.* $B = E_+ \cap H$, subject to the additional requirement that $E_+ = \bigcup_{\alpha \geq 0} \alpha B$. A base-norm space is then directed ordered vector space (E, E_+) and a choice of base B for E_+, such that

$$U = \mathrm{co}(B \cup -B)$$

is radially compact.

Proposition 2.2.18.

(i) The base of a base-norm space in the Alfsen-Shultz sense cannot be empty.

(ii) The base-norm space (in our sense) $(\{0\}, \{0\}, 0)$*, is not a base-norm space in the Alfsen-Shultz sense.*

Proof.

(i) Suppose $B = \emptyset$. Then $E_+ = \bigcup_{\alpha \geq 0} \alpha B = \emptyset$, and so $E = E_+ - E_+ = \emptyset$, which contradicts E being a vector space, as it must contain 0.

(ii) It is clear that since the base of $\{0\}$ is empty it cannot be an Alfsen-Shultz base-norm space with the same base. In fact, it cannot be one at all, because it contains no hyperplanes. □

Proposition 2.2.19. *Every non-zero radially compact pre-base-norm space is an Alfsen-Shultz base-norm space, with the same base and unit ball.*

Proof. By definition, $\tau^{-1}(1)$ is a hyperplane, and $B = E_+ \cap \tau^{-1}(1)$. Since $\tau(0) = 0$, we have $0 \notin \tau^{-1}(1)$. Since $\mathrm{co}(B \cup -B)$ is radially compact, we only need to show that B is actually a base for E_+, *i.e.* that $E_+ = \bigcup_{\alpha \geq 0} \alpha B$.

Let $x \in E_+$. We start with the case that $x = 0$. Since $E \neq 0$, we have that there is $x' \in B$ (Lemma 2.2.1), and therefore $0 \cdot x' = x \in 0 \cdot B$. If $x \neq 0$, using strict positivity (Lemma 2.2.2) we have $\tau(x) > 0$. Therefore we can define $x' = \frac{x}{\tau(x)}$, and $\tau(x') = 1$, so $x' \in B$, and therefore $x \in \tau(x)B$. We have proven that $E_+ \subseteq \bigcup_{\alpha \geq 0} \alpha B$. The opposite inclusion follows from the fact that E_+ is a cone and $B \subseteq E_+$. □

Proposition 2.2.20. *Every base-norm space in the Alfsen-Shultz sense is a radially compact base-norm space, with the same base and unit ball.*

Proof. We must define τ. We have, by Proposition 2.2.18 (i) that the base B is not empty, so there is $x \in B$. By taking the hyperplane H, and producing $H - x$, we have a hyperplane passing through 0, and therefore $E/(H - x) \cong \mathbb{R}$, so we can define a map $\tau' : E \to \mathbb{R}$ by composing the surjection $E \to E/(H - x)$ with the isomorphism with $\mathbb{R}$. If $\tau'(x) = 0$, then $0 - x \in H - x$ and hence $0 \in H$, a contradiction, so $\tau'(x) \neq 0$. We can therefore take $\tau = \frac{\tau'}{\tau'(x)}$. We have shown that τ is not zero.

We next show that $B = E_+ \cap \tau^{-1}(1)$, and prove that τ is positive last. If $y \in E_+$ and $\tau(y) = 1$, then $\tau(y - x) = 0$, so $y - x \in H - x$, therefore $y \in H$, and so $y \in H \cap E_+ = B$. This shows $E_+ \cap \tau^{-1}(1) \subseteq B$. For the opposite inclusion, if $y \in B$, then $y \in H$ and so $y - x \in H - x$, meaning $\tau(y - x) = 0$. Therefore $\tau(y) = \tau(x) = 1$, and therefore $y \in E_+ \cap \tau^{-1}(1)$.

To show that τ is positive, suppose $y \in E_+$. Because B is a base, there exists some $\alpha \geq 0$ such that $y \in \alpha B$. This means that there is some $y' \in B$ such that $y = \alpha y'$. We have

$$\tau(y) = \tau(\alpha y') = \alpha \tau(y') = \alpha \geq 0,$$

using the previous result that $B = E_+ \cap \tau^{-1}(1)$ to make the penultimate step.

Since B is non-empty, the radial compactness of $\mathrm{co}(B \cup -B)$ implies that of $\mathrm{absco}(B)$, as they are equal by Lemma 0.1.1. We have therefore shown that (E, E_+, τ) is a radially compact pre-base-norm space. It is therefore a base-norm space by Proposition 2.2.6 (ii). □

All together, this shows that, except for the zero base-norm space, Alfsen and Shultz's definition of a base-norm space is at least as strict as ours, because it coincides with radially compact base-norm spaces. In the appendix we give a counterexample (Counterexample A.6.2) due to Asimow but published by Ellis [36] of a Banach base-norm space such that $U = \mathrm{absco}(B)$ is not radially compact. Therefore, for nonzero vector spaces, Alfsen and Shultz's definition is stricter.

2.3 Relationship to C* and W*-algebras

In Proposition 1.2.10 we saw that taking the self-adjoint part of a C*-algebra yields a full and faithful functor to the category of Banach order-unit spaces. This, in fact, is one of the motivations for the definition of an order-unit space. In this section, we describe another full and faithful functor and the kind of space that motivated the definition of a base-norm space.

A W*-algebra is a C*-algebra A that is isomorphic to the dual space of some Banach space A_* [117, Definition 1.1.2][128, Theorem 3.5]. Equivalently, it is a C*-algebra A such that there exists a Banach space A_* and a duality $\langle \text{-}, \text{-} \rangle : A \times A_* \to \mathbb{C}$ such that the map $A \to (A_*)^*$ defined by the duality is an isometry. W*-algebras were defined by Sakai to give a characterization of the C*-algebras arising from von Neumann algebras up to isomorphism. The space A_* is called the *predual* and is unique up to isomorphism [117, Corollary 1.13.3][128, Corollary 3.9]. The urexample of a W*-algebra and its predual are $B(\mathcal{H})$, the C*-algebra of all bounded operators on a Hilbert space $\mathcal{H}$, and its predual $\mathcal{TC}(\mathcal{H})$, the space of trace-class operators, the pairing being

$$\langle a, \rho \rangle = \mathrm{tr}(a\rho),$$

where $a \in B(\mathcal{H})$ and $\rho \in \mathcal{TC}(\mathcal{H})$[2]. In this case, one can define self-adjoint and positive elements of $\mathcal{TC}(\mathcal{H})$ in the usual way as $\mathcal{TC}(\mathcal{H}) \subseteq B(\mathcal{H})$, and the trace $\tau(\rho)$ of a trace-class operator ρ can be defined as the sum of the diagonal entries of a matrix for ρ, expressed in some orthonormal basis[3]. The convex set

$$\mathcal{DM}(\mathcal{H}) = \{\rho \in \mathcal{TC}(\mathcal{H}) \mid \rho \text{ positive and } \tau(\rho) = 1\},$$

is known as the set of density matrices. So we have the ingredients for a base-norm space with base $\mathcal{DM}(\mathcal{H})$.

[2]In the finite dimensional case, every operator is trace-class.

[3]Independence of the orthonormal basis chosen is essentially what being of trace-class amounts to.

We first discuss (continuous) linear functionals on a C*-algebra A, or elements of A^*. The involution $\text{-}^* : A \to \overline{A}$ can be used to define an involution on A^*:

$$\phi^*(a) = \overline{\phi(a^*)},$$

where $a \in A$ and $\phi \in A^*$[29, §1.1.10]. A functional $\phi : A \to \mathbb{C}$ is therefore *self-adjoint* if $\phi^* = \phi$, equivalently, if $\phi(a^*) = \overline{\phi(a^*)}$, *i.e.* ϕ preserves the -^* operation.

In the case of a general W*-algebra A, we can embed the predual A_* isometrically into A^* by using the other map derived from the pairing. We can use the freedom of choosing the predual up to isomorphism to redefine it to be this subset of A^*. It can equivalently be defined, by Proposition 0.3.2 to be the elements of A^* that are $\sigma(A, A_*)$-continuous. These are known as *normal linear functionals*, and the $\sigma(A, A_*)$ topology is called the *ultraweak* or *σ-weak* topology. One can therefore define $\text{SA}(A_*)$ to be the $\mathbb{R}$-vector space of self-adjoint elements of A_*, and A_{*+} to be the set of positive elements of A_*. We have a linear map $\tau : \text{SA}(A_*) \to \mathbb{R}$ defined as $\tau(\phi) = \phi(1)$.

Theorem 2.3.1. *If A_* is the predual of a W*-algebra,* $(\text{SA}(A_*), A_{*+}, \tau)$ *is a (radially compact) Banach base-norm space. If* **Predual** *is the category having preduals as objects and linear, positive, trace-preserving maps, restriction of morphisms defines a full and faithful functor* $\text{SA} : \mathbf{Predual} \to \mathbf{BBNS}$.

Proof. The fact that the self-adjoint part of the predual of a W*-algebra is proven in [34, Proposition 5.1] and in [6, Corollary 2.96]. The definitions of W*-algebra and base-norm space used in those references exclude the W*-algebra in which $0 = 1$, but the self-adjoint part of the predual is the unique base-norm space with empty base in this case, so there is no problem. Note that this implies that the real span of A_{*+} is $\text{SA}(A_*)$, so if $f : A_* \to B_*$ is a map of preduals, then if $\phi \in \text{SA}(A_*)$, we have $f(\phi) \in \text{SA}(B_*)$. Preservation of identity and composition for the functor SA is then trivial. Analogously to Lemma 1.2.2, elements of the predual have a decomposition into real and imaginary self-adjoint parts with

$$\phi_{\Re} = \frac{\phi + \phi^*}{2} \qquad\qquad \phi_{\Im} = \frac{\phi - \phi^*}{2i}$$

and the proof of fullness and faithfulness proceeds along the same lines as Proposition 1.2.10 so is omitted. □

The base of $\text{SA}(A_*)$ is the set of states that are $\sigma(A, A*)$ continuous as maps $A \to \mathbb{C}$, and is accordingly known as the set of *normal states*. In the special case that $A = B(\mathcal{H})$, this is $\mathcal{DM}(\mathcal{H})$, as would be expected.

In the next chapter, we will prove a statement implying that the "predual" of any order-unit space is a base-norm space, giving a proof that the self-adjoint part of the predual of a W*-algebra is a base-norm space independently of the results cited above.

2.4 Relationship to Monads

The monads are $\mathcal{D}, \mathcal{D}^{\leq 1}, \mathcal{D}_\infty$ and $\mathcal{D}_\infty^{\leq 1}$, all functors $\mathbf{Set} \to \mathbf{Set}$. The monad $\mathcal{D}$ is the usual distribution monad, $\mathcal{D}^{\leq 1}$ the subnormalized version, $\mathcal{D}_\infty$ the infinite distribution monad, and $\mathcal{D}_\infty^{\leq 1}$ its subnormalized version. We summarize the definitions here, but do not prove they are monads as that is adequately explained elsewhere. Apparently the idea of using infinite convex combinations on state spaces is due to M. A. Gerzon [34, p. 214], and later appeared under the name *superconvex sets* [113], see also [76] and [77].

On objects, the functors are defined:

$$\begin{aligned}
\mathcal{D}(X) &= \left\{ \phi : X \to [0,1] \,\middle|\, \mathrm{supp}(\phi) \text{ finite and } \sum_{x \in X} \phi(x) = 1 \right\} \\
\mathcal{D}^{\leq 1}(X) &= \left\{ \phi : X \to [0,1] \,\middle|\, \mathrm{supp}(\phi) \text{ finite and } \sum_{x \in X} \phi(x) \leq 1 \right\} \\
\mathcal{D}_\infty(X) &= \left\{ \phi : X \to [0,1] \,\middle|\, \sum_{x \in X} \phi(x) = 1 \right\} \\
\mathcal{D}_\infty^{\leq 1}(X) &= \left\{ \phi : X \to [0,1] \,\middle|\, \sum_{x \in X} \phi(x) \leq 1 \right\}.
\end{aligned}$$

On a map $f : X \to Y$ in $\mathbf{Set}$, we give the formula for $\mathcal{D}$ only, as it is the same for the other three. Let $\phi \in \mathcal{D}(X)$ and $y \in Y$:

$$\mathcal{D}(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x).$$

The unit and counit are defined the same for all four monads, so we give the definition only for $\mathcal{D}$:

$$\begin{aligned}
\eta_X &: X \to \mathcal{D}(X) \\
\eta_X(x)(x') &= 1 \text{ if } x = x' \\
\eta_X(x)(x') &= 0 \text{ otherwise} \\
\mu_X &: \mathcal{D}^2(X) \to \mathcal{D}(X) \\
\mu_X(\Psi)(x) &= \sum_{\phi \in \mathcal{D}(X)} \Psi(\phi) \cdot \phi(x).
\end{aligned}$$

There are monad morphisms $\tau : \mathcal{D} \Rightarrow \mathcal{D}_\infty$ and $\tau^{\leq 1} : \mathcal{D}^{\leq 1} \Rightarrow \mathcal{D}_\infty^{\leq 1}$.

Proposition 2.4.1. *The family of maps $\tau_X : \mathcal{D}(X) \to \mathcal{D}_\infty(X)$ taking the finite distributions into the infinite ones is natural and a monad morphism. The same is true for $\tau_X^{\leq 1} : \mathcal{D}^{\leq 1}(X) \to \mathcal{D}_\infty^{\leq 1}(X)$.*

Proof. The definition of τ_X is

$$\tau_X(\phi) = \phi. \tag{2.1}$$

This is clearly natural. The definition of $\tau_X^{\leq 1}$ is identical.

In the following we only give the proof for τ_X as the proof for $\tau_X^{\leq 1}$ is identical as the definitions of the maps involved coincide.

The triangle

$$\begin{array}{ccc} I & \xrightarrow{\eta^{\mathcal{D}}} & \mathcal{D} \\ & {\scriptstyle \eta^{\mathcal{D}}_{\infty}} \searrow & \downarrow{\scriptstyle \tau} \\ & & \mathcal{D}_\infty \end{array}$$

commutes as $\eta_X^{\mathcal{D}_\infty}(x) \in \mathcal{D}(X)$ and τ_X is just the inclusion morphism. The pentagon

$$\begin{array}{ccc} \mathcal{D}^2 & \xrightarrow{\mathcal{D}\tau} & \mathcal{D}\mathcal{D}_\infty \\ {\scriptstyle \mu^{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \tau\mathcal{D}_\infty} \\ \mathcal{D} & & \mathcal{D}_\infty^2 \\ & {\scriptstyle \tau}\searrow & \downarrow{\scriptstyle \mu^{\mathcal{D}}_{\infty}} \\ & & \mathcal{D}_\infty \end{array}$$

can be proved to commute as follows. Let $\Phi \in \mathcal{D}^2(X)$, and $x \in X$. For the lower left path we have

$$\begin{aligned} \tau_X(\mu_X^{\mathcal{D}}(\Phi))(x) &= \mu_X^{\mathcal{D}}(\Phi)(x) \\ &= \sum_{\psi \in \mathcal{D}(X)} \Phi(\psi)\psi(x) \end{aligned}$$

and for the upper right path we have

$$\begin{aligned} \mu_X^{\mathcal{D}_\infty}(\tau_{\mathcal{D}_\infty(X)}(\mathcal{D}(\tau_X)(\Phi)))(x) &= \sum_{\psi \in \mathcal{D}_\infty(X)} \tau_{\mathcal{D}_\infty(X)}(\mathcal{D}(\tau_X)(\Phi))(\psi) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_\infty(X)} \mathcal{D}(\tau_X)(\Phi)(\psi) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_\infty(X)} \left(\sum_{\psi' \in \tau_X^{-1}(\psi)} \Phi(\psi) \right) \cdot \psi(x) \\ &= \sum_{\psi \in \mathcal{D}_\infty(X)} \sum_{\psi' \in \tau_X^{-1}(\psi)} \Phi(\psi) \cdot \psi(x) \end{aligned}$$

The inner sum consists of one term if $\psi \in \mathcal{D}(X)$, and is zero if ψ has infinite support. Therefore we have

$$= \sum_{\psi \in \mathcal{D}_\infty(X)} \Phi(\psi)\psi(x)$$

also for the top right path and the diagram commutes. □

We therefore have forgetful functors $\mathcal{EM}(\mathcal{D}_\infty) \to \mathcal{EM}(\mathcal{D})$ and $\mathcal{EM}(\mathcal{D}^{\leq 1}) \to \mathcal{EM}(\mathcal{D}_\infty^{\leq 1})$.

2.4.1 The Base and Subbase Functors and Their Left Adjoints

We saw earlier what the base B_E of a pre-base-norm space (E, E_+, τ) is and the subbase $B_E^{\leq 1}$. We can define two functors

$$B_{\mathbf{Set}} : \mathbf{PreBNS} \to \mathbf{Set} \qquad\qquad B_{\mathbf{Set}}^{\leq 1} : \mathbf{PreBNS}_{\leq 1} \to \mathbf{Set},$$

on objects being the base and the subbase. On maps, these are simply restriction, which is well defined by Lemmas 2.2.7 and 2.2.10 respectively. By restriction to the full subcategories **BBNS** and $\mathbf{BBNS}_{\leq 1}$ we also have functors from those categories to **Set**. We now define their left adjoints $\ell^1_c : \mathbf{Set} \to \mathbf{PreBNS}$ and $\ell^1 : \mathbf{Set} \to \mathbf{BBNS}$. We define these on a set X, as

$$\ell^1_c(X) = \{\phi : X \to \mathbb{R} \mid \operatorname{supp}(\phi) \text{ is finite}\}$$
$$\ell^1(X) = \left\{\phi : X \to \mathbb{R} \,\middle|\, \sum_{x \in X} |\phi(x)| < \infty \right\}.$$

The vector space structure is defined pointwise, and it is clear that $\ell^1_c(X)$ is a subspace of $\ell^1(X)$. We define the positive cone in each to be those ϕ such that $\phi(x) \geq 0$ for all $x \in X$, and we define the trace to be

$$\tau(\phi) = \sum_{x \in X} \phi(x),$$

which exists for all $\phi \in \ell^1(X)$ by Lemma 0.1.11.

It will be useful later to observe that for each $\phi \in \ell^1(X)$, we can separate it into its positive and negative parts:

$$\phi_+(x) = \begin{cases} \phi(x) & \text{if } \phi(x) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \phi_-(x) = \begin{cases} -\phi(x) & \text{if } \phi(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

We have that $\phi_+, \phi_- \in \ell^1(X)$, and $\phi_+, \phi_- \in \ell^1_c(X)$ if $\phi \in \ell^1_c(X)$, and $\phi = \phi_+ - \phi_-$.

It is a standard fact that $\ell^1(X)$ is a Banach space with the norm

$$\|\phi\| = \sum_{x \in X} |\phi(x)|,$$

see [24, Example 1.9]

Lemma 2.4.2. *The space $\ell^1_c(X)$ is dense in $\ell^1(X)$.*

Proof. Let $\psi \in \ell^1(X)$. We want to show that for all $\epsilon > 0$, there is a $\phi_\epsilon \in \ell^1_c(X)$ such that $\|\psi - \phi\| < \epsilon$. If ψ has finite support then we can simply take $\phi = \psi$, so we now reduce to the case that ψ has infinite (hence countable by Lemma 0.1.9) support, which we enumerate as a sequence $(x_i)_{i\in\mathbb{N}}$. Let $\epsilon > 0$. Since ψ is absolutely summable, there is an $N \in \mathbb{N}$ such that $\left|\sum_{i=1}^{\infty} |\psi(x_i)| - \sum_{i=1}^{N} |\psi(x_i)|\right| < \epsilon$. Define

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x = x_i \text{ for some } 0 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}.$$

We can now see that

$$\begin{aligned} \|\psi - \phi\| &= \sum_{x\in X} |(\psi - \phi)(x)| \\ &= \sum_{i=1}^{N} |\psi(x_i) - \phi(x_i)| + \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| \\ &= \sum_{i=1}^{N} 0 + \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| \\ &= \sum_{i=N+1}^{\infty} |\psi(x_i) - 0| < \epsilon. \end{aligned}$$

□

Proposition 2.4.3. *With the above definitions, $(\ell^1_c(X), \ell^1_c(X)_+, \tau)$ is a radially compact base-norm space and $(\ell^1(X), \ell^1(X)_+, \tau)$ is a radially compact Banach base-norm space for any set X.*

Proof. The fact that $\phi = \phi_+ - \phi_-$ implies that $\ell^1_c(X)$ and $\ell^1(X)$ are generated by their positive cones.

If $\ell^1_c(X)$ is not 0, then since $\ell^1_c(\emptyset) = 0$, we have that $X \neq \emptyset$ and so there is some $y \in X$. The function δ_y defined by

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is an element of $\ell^1_c(X)$. Then

$$\tau(\delta_y) = \sum_{x\in X} \delta_y(x) = 1$$

so τ is not the 0 map on $\ell^1_c(X)$, and therefore not on $\ell^1(X)$ either.

The last part to prove is that, with B being the base, absco(B) is radially compact, as radially compact pre-base-norm spaces are base-norm spaces (Proposition 2.2.6). We show this in $\ell^1(X)$ first by showing that absco(B) is equal to the closed unit ball of the usual norm:

$$U = \{\phi \in \ell^1(X) \mid \|\phi\| \leq 1\} = \left\{\phi \in \ell^1(X) \,\middle|\, \sum_{x \in X} |\phi(x)| \right\}.$$

We dispose of the trivial case first. If $\ell^1(X) = 0$, then $U = \{0\} = \text{absco}(B)$. Now we assume that $\ell^1(X) \neq 0$ and so $\text{absco}(B) = \text{co}(-B \cup B)$.

- $\text{absco}(B) \subseteq U$: If $\phi \in \text{co}(-B \cup B)$, it can be expressed as $\alpha\phi_+ + (1-\alpha)\phi_-$ where $\phi_+ \in B$ and $\phi_- \in -B$ and $\alpha \in [0,1]$. Because U, being a unit ball, is absolutely convex, it suffices to show that $B \subseteq U$ to show that any expression $\alpha\phi_+ + (1-\alpha)\phi_- \in U$. If $\phi \in B$, we have that

$$\begin{aligned} \|\phi\| &= \sum_{x \in X} |\phi(x)| \\ &= \sum_{x \in X} \phi(x) && \phi \in \ell^1(X)_+ \\ &= \tau(\phi) = 1 && \phi \in B, \end{aligned}$$

  so $\phi \in U$. Therefore $\text{absco}(B)$ is radially bounded, and so $\ell^1(X)$ is a pre-base-norm space.

- $U \subseteq \text{absco}(B)$: Let $\phi \in U$. We first define $|\phi| = \phi_+ + \phi_-$ and observe that $|\phi|(x) = |\phi(x)|$. By assumption

$$\begin{aligned} 1 \geq \|\phi\| &= \sum_{x \in X} |\phi(x)| = \sum_{x \in X} |\phi|(x) = \sum_{x \in X} \phi_+(x) + \sum_{x \in X} \phi_-(x) \\ &= \tau(\phi_+) + \tau(\phi_-) \end{aligned}$$

  We now have four cases:

  - $\tau(\phi_+) = \tau(\phi_-) = 0$: By the strict positivity of τ (Lemma 2.2.2), we have $\phi_+ = \phi_- = 0$, so $\phi = 0$ and $\phi \in \text{absco}(B)$ because all absolutely convex sets contain zero.
  - $\tau(\phi_+) \neq 0$ but $\tau(\phi_-) = 0$: Then $\phi = \phi_+$, and $\tau(\phi)$ is invertible, so $\tau(\phi)^{-1}\phi \in B$, and $\tau(\phi) \leq 1$ implies that the absolutely convex combination $\tau(\phi)(\tau(\phi^{-1})\phi) = \phi \in \text{absco}(B)$.
  - $\tau(\phi_-) \neq 0$ but $\tau(\phi_-) = 0$: This case is similar to the previous one.
  - $\tau(\phi_+) \neq 0$ and $\tau(\phi_-) \neq 0$: Define $\phi'_\pm = \tau(\phi_\pm)^{-1}\phi_\pm$ in each case. Then $\phi'_\pm \in B$ and $\tau(\phi_+)\phi'_+ - \tau(\phi_-)\phi_- = \phi$. As $\tau(\phi_+) + \tau(\phi_-) \leq 1$, this is an absolutely convex combination and shows that $\phi \in \text{absco}(B)$.

Radial compactness follows because the closed unit ball intersecting any ray is a closed bounded subset of that ray and hence compact. Any ray in $\ell^1_c(X)$ is also a ray in $\ell^1(X)$ so $\ell^1_c(X)$ has radially compact unit ball too. Finally, $\ell^1(X)$

is a Banach base-norm space because it is a Banach space in its usual norm and the base norm coincides with the usual norm because the closed unit balls are the same for each. □

On maps, we define for $f : X \to Y$ in **Set**:

$$\ell^1(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x),$$

where $\phi \in \ell^1(X)$ and $y \in Y$. Each of these sums is absolutely convergent since it has as subset of the terms of an absolutely convergent sum, so this is well-defined. We define $\ell^1_c(f)$ in the same manner, restricting $\ell^1(f)$ to $\ell^1(X)$. Since for each $\phi \in \ell^1(X)$ only finitely many values of x have $\phi(x) \neq 0$, this is also true for $\ell^1(f)(\phi)$, so the above definition has the correct type.

Proposition 2.4.4. *As defined, ℓ^1_c is a functor* **Set** $\to$ **BNS** *and ℓ^1 is a functor* **Set** $\to$ **BBNS**.

Proof. The proof proceeds by showing this for ℓ^1 first and deducing that it is so for ℓ^1_c afterwards.

Let $f : X \to Y$ be a function. We must first show that $\ell^1(f)$ is a positive trace-preserving map. For linearity, consider $\alpha, \beta \in \mathbb{R}$ and $\phi, \psi \in \ell^1(X)$, and $y \in Y$. Now

$$\begin{aligned}
\ell^1(f)(\alpha\phi + \beta\psi)(y) &= \sum_{x \in f^{-1}(y)} (\alpha\phi + \beta\psi)(x) \\
&= \sum_{x \in f^{-1}(y)} (\alpha\phi(x) + \beta\psi(x)) \\
&= \alpha \sum_{x \in f^{-1}(y)} \phi(x) + \beta \sum_{x \in f^{-1}(y)} \psi(x) \\
&= \alpha\ell^1(f)(\phi)(y) + \beta\ell^1(f)(\psi)(y) \\
&= (\alpha\ell^1(f)(\phi) + \beta\ell^1(f)(\psi))(y).
\end{aligned}$$

To show positivity, suppose $\phi \in \ell^1(X)_+$. Then

$$\ell^1(f)(\phi)(y) = \sum_{x \in f^{-1}(y)} \phi(x).$$

As the sum of nonnegative numbers, this is a nonnegative number, so $\ell^1(f)(\phi) \in \ell^1(Y)_+$.

To show that $\ell^1(f)$ is trace-preserving, let τ denote the trace of $\ell^1(X)$ and σ that of $\ell^1(Y)$. We want to show $\sigma \circ \ell^1(f) = \tau$. We start with $\phi \in \ell^1(X)$:

$$\sigma(\ell^1(f)(\phi)) = \sum_{y \in Y} \ell^1(f)(\phi)(y) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) = \sum_{x \in X} \phi(x) = \tau(\phi).$$

Since $\ell^1_c(f)$ is the restriction of $\ell^1(f)$ to $\ell^1(X)$, it is also linear, positive and trace preserving and so defines a **BNS** map $\ell^1_c(X) \to \ell^1_c(Y)$.

We must now show that ℓ^1 is functorial, *i.e.* that it preserves identity maps and composition. To show the preservation of identity maps, consider $\mathrm{id}_X : X \to X$ for an arbitrary set X, and let $\phi \in \ell^1(X)$ and $x \in X$. Then

$$\ell^1(\mathrm{id}_X)(\phi)(x) = \sum_{x \in \mathrm{id}_X^{-1}(x)} \phi(x) = \phi(x),$$

therefore $\ell^1(\mathrm{id}_X)(\phi) = \phi$ and so $\ell^1(\mathrm{id}_X) = \mathrm{id}_{\ell^1(X)}$.

To show that ℓ^1 preserves composition, consider three sets X, Y and Z, and two functions $f : X \to Y$ and $Y \to Z$, and let $\phi \in \ell^1(X)$ and $z \in Z$. Then

$$\begin{aligned}
(\ell^1(g) \circ \ell^1(f))(\phi)(z) &= \ell^1(g)(\ell^1(f)(\phi))(z) \\
&= \sum_{y \in g^{-1}(z)} \ell^1(f)(\phi)(y) \\
&= \sum_{y \in g^{-1}(z)} \sum_{x \in f^{-1}(y)} \phi(x) \\
&= \sum_{x \in f^{-1}(g^{-1}(z))} \phi(x) \\
&= \sum_{x \in (g \circ f)^{-1}(z)} \phi(x) \\
&= \ell^1(g \circ f)(\phi)(z),
\end{aligned}$$

applying functional extensionality twice, we get the required functoriality $\ell^1(g) \circ \ell^1(f) = \ell^1(g \circ f)$. As ℓ^1_c is defined by restricting ℓ^1, ℓ^1_c is also a functor. □

The functor ℓ^1_c can also be composed with the inclusion $\mathbf{BNS} \hookrightarrow \mathbf{BNS}_{\leq 1}$ to get a functor $\mathbf{Set} \to \mathbf{BNS}_{\leq 1}$, and similarly for ℓ^1, and in fact ℓ^1_c can also be composed with the inclusion $\mathbf{BNS} \hookrightarrow \mathbf{PreBNS}$.

The following theorem is the analogue for base-norm spaces of what Pumplün and Röhrl prove for the unit ball functor on normed spaces [107].

Theorem 2.4.5. $\ell^1_c : \mathbf{Set} \to \mathbf{PreBNS}$ *is left adjoint to* $B_{\mathbf{Set}}$ *and* $B^{\leq 1}_{\mathbf{Set}}$*, and* $\ell^1 : \mathbf{Set} \to \mathbf{BBNS}$ *is left adjoint to* $B_{\mathbf{Set}}$ *and* $B^{\leq 1}_{\mathbf{Set}}$ *when restricted to Banach base-norm spaces.*

Proof. In general we work with ℓ^1 as this is the more difficult case, showing how the ℓ^1_c case differs when necessary.

We use a unit and its universal property to define the adjunctions (Theorem 0.4.1 (ii)). We define the units as follows, for X a set and $x, x' \in X$:

$$\eta_X : X \to B_{\mathbf{Set}}(\ell^1(X)) \qquad \eta^{\leq 1}_X : X \to B^{\leq 1}_{\mathbf{Set}}(\ell^1(X))$$

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \qquad \eta^{\leq 1}_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

We show that $\eta_X(x) \in B_{\mathbf{Set}}(\ell^1_c(X))$. This implies $\eta^{\leq 1}_X(x) \in B^{\leq 1}_{\mathbf{Set}}(\ell^1_c(X))$ as the definition is the same, and that $\eta_X(x) \in B_{\mathbf{Set}}(\ell^1(X))$ and $\eta^{\leq 1}_X(x) \in B^{\leq 1}_{\mathbf{Set}}(\ell^1(X))$. First, observe that $\eta_X(x)(x')$ has finite support and is only 1 or 0 so is in $\ell^1_c(X)_+$. Secondly, taking the trace

$$\tau(\eta_X(x)) = \sum_{x' \in X} \eta_X(x)(x') = 1,$$

which shows $\eta_X(x) \in B_{\mathbf{Set}}(\ell^1_c(X))$.

To show that η_X and $\eta^{\leq 1}_X$ are natural, we again show only the proof for η_X, as the proof for $\eta^{\leq 1}_X$ is essentially identical. We want to show that for any function $f : X \to Y$,

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B_{\mathbf{Set}}(\ell^1(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle B_{\mathbf{Set}}(\ell^1(f))} \\ Y & \xrightarrow[\eta_Y]{} & B_{\mathbf{Set}}(\ell^1(Y)) \end{array}$$

commutes, *i.e.* that $B_{\mathbf{Set}}(\ell^1(f)) \circ \eta_X = \eta_Y \circ f$. So let $x \in X$ and $y \in Y$. For the lower left path we have that $\eta_Y(f(x))(y)$ is 1 if $f(x) = y$ and 0 otherwise. For the upper right path we have

$$B_{\mathbf{Set}}(\ell^1(f))(\eta_X(x))(y) = \sum_{x' \in f^{-1}(y)} \eta(x)(x').$$

The right hand side is 1 only if $x \in f^{-1}(y)$, otherwise it is 0. In other words, it is 1 if $f(x) = y$ and 0 otherwise. Therefore the two paths are equal by functional extensionality.

We now prove the universal property. In the following, we do the η_X case in full, and the $\eta^{\leq 1}_X$ case only when it differs (the unique map need only be trace-reducing, not trace preserving). We will also only give the ℓ^1 case in full, as the ℓ^1_c case is mostly a restriction of it. We want to show that for every set X and Banach base-norm space (E, E_+, σ), given a function $f : X \to B_{\mathbf{Set}}(E)$, there is a unique $g \in \mathbf{BBNS}(\ell^1(X), E)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B_{\mathbf{Set}}(\ell^1(X)) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle B_{\mathbf{Set}}(g)} \\ & & B_{\mathbf{Set}}(E). \end{array} \tag{2.2}$$

In the ℓ^1_c case we only assume that E is a pre-base-norm space in the above, not necessarily a Banach base-norm space.

We define g as follows (in both cases) for $\phi \in \ell^1(X)$ as:

$$g(\phi) = \sum_{x \in X} \phi(x) f(x).$$

We first need to show that $g(\phi)$ defines an element of E. We have that $\|\phi(x)f(x)\| = |\phi(x)|\|f(x)\| \leq \phi(x)$ because $f(x) \in B^{\leq 1}_{\mathbf{Set}}$ and $B^{\leq 1}_{\mathbf{Set}}$ is a subset of the unit ball of E. So by Lemma A.1.2

$$\sum_{x \in X} \|\phi(x)f(x)\| \leq \sum_{x \in X} |\phi(x)|.$$

Therefore $(\phi(x)f(x))_{x \in X}$ is an absolutely summable family in E, a Banach space, so its sum converges by Lemma 0.1.11. In the ℓ^1_c case, the sum is finite and so the previous step is not necessary.

To show g is linear, let $\alpha, \beta \in \mathbb{R}$ and $\phi, \psi \in \ell^1(X)$. As in the previous part of the proof, we can see that $\sum_{x \in X} \alpha\phi(x)f(x)$ and $\sum_{x \in X} \beta\psi(x)f(x)$ are absolutely convergent in E. We can therefore apply Lemma 0.1.12 to conclude that $\sum_{x \in X}(\alpha\psi + \beta\psi)(x)f(x) = \sum_{x \in X} \alpha\psi(x)f(x) + \sum_{x \in X} \beta\phi(x)f(x)$, and that the former sum converges. Then:

$$\begin{aligned} g(\alpha\psi + \beta\phi) &= \sum_{x \in X} (\alpha\psi + \beta\phi)(x)f(x) \\ &= \sum_{x \in X} \alpha\psi(x)f(x) + \sum_{x \in X} \beta\phi(x)f(x) \\ &= \alpha \sum_{x \in X} \psi(x)f(x) + \beta \sum_{x \in X} \phi(x)f(x) \\ &= \alpha g(\psi) + \beta g(\phi). \end{aligned}$$

Now, if $\phi \in \ell^1(X)_+$, then

$$g(\phi) = \sum_{x \in X} \phi(x)f(x),$$

and each $\phi(x)f(x) \in E_+$, as E_+ is a cone, and the partial sums are in E_+ for the same reason. Since E_+ is closed, $g(\phi) \in E_+$, and g is a positive map. In the ℓ^1_c case, we only assume that E is a pre-base-norm space, so we do not have that E_+ is closed. However, in this case the sum is finite and so is an element of E_+ simply because it is a cone.

- If $f : X \to B_{\mathbf{Set}}(E)$, g is trace-preserving:

  We want to show that $\sigma \circ g = \tau$, where τ is the trace of $\ell^1(X)$ and σ that of E. Let $\phi \in \ell^1(X)$. We have

$$\begin{aligned}
\tau(g(\phi)) &= \tau\left(\sum_{x\in X}\phi(x)f(x)\right)\\
&= \tau\left(\lim_{j\in\mathcal{P}_{\mathrm{fin}}(X)}\sum_{x\in j}\phi(x)f(x)\right)\\
&= \lim_{j\in\mathcal{P}_{\mathrm{fin}}(X)}\sum_{x\in j}\phi(x)\tau(f(x)) && \tau \text{ continuous and linear}\\
&= \sum_{x\in X}\phi(x)\tau(f(x))\\
&= \sum_{x\in X}\phi(x) = \sigma(\phi).
\end{aligned}$$

- If $f : X \to B^{\leq 1}_{\mathbf{Set}}(E)$, g is trace-reducing:

  Let $\phi \in \ell^1(X)_+$. We want to show that $\tau(g(\phi)) \leq \sigma(\phi)$. So

$$\begin{aligned}
\tau(g(\phi)) &= \sum_{x\in X}\phi(x)\tau(f(x)) && \text{by previous proof}\\
&\leq \sum_{x\in X}\phi(x) && \text{since } f(x)\in B^{\leq 1}_{\mathbf{Set}}(E)\\
&= \sigma(\phi).
\end{aligned}$$

We now show that $B_{\mathbf{Set}}(g)$ and $B^{\leq 1}_{\mathbf{Set}}(g)$ make their respective versions of (2.2) commute and that g is the unique such map. The proofs of the $B_{\mathbf{Set}}$ and $B^{\leq 1}_{\mathbf{Set}}$ cases look identical, so we only give the proof for $B_{\mathbf{Set}}(g)$. If $x \in X$, then

$$B_{\mathbf{Set}}(g)(\eta_X(x)) = g(\eta_X(x)) = g(\eta_X(x)) = \sum_{x'\in X}\eta_X(x)(x')f(x') = f(x).$$

Finally we show the uniqueness. Suppose $B_{\mathbf{Set}}(h)$ makes (2.2) commute in place of $B_{\mathbf{Set}}(g)$, *i.e.* $B_{\mathbf{Set}}(h)\circ\eta_X = f$. If $\psi \in \ell^1_c(X)$, we have

$$\psi = \sum_{i=1}^{n}\psi(x_i)\eta_X(x_i)$$

where the elements $x_i \in X$ are an enumeration of the support of ψ. This implies that

$$\begin{aligned}
h(\psi) &= h\left(\sum_{i=1}^{n}\psi(x_i)\eta_X(x_i)\right) = \sum_{i=1}^{n}\psi(x_i)h(\eta_X(x_i)) \\
&= \sum_{i=1}^{n}\psi(x_i)B_{\mathbf{Set}}(h)(\eta_X(x_i)) = \sum_{i=1}^{n}\psi(x_i)f(x_i) \\
&= \sum_{i=1}^{n}\psi(x_i)B_{\mathbf{Set}}(g)(\eta_X(x_i)) = \sum_{i=1}^{n}\psi(x_i)g(\eta_X(x_i)) \\
&= g\left(\sum_{i=1}^{n}\psi(x_i)\eta_X(x_i)\right) = g(\psi).
\end{aligned}$$

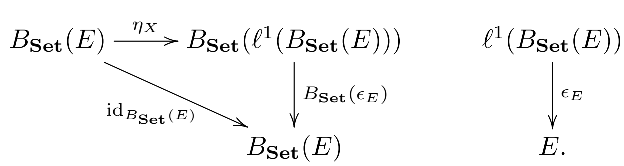
We have therefore finished in the ℓ^1_c case. In the ℓ^1 case, we use the fact that $\ell^1_c(X)$ is dense in $\ell^1(X)$ (Lemma 2.4.2) and that g, h are continuous maps (Proposition 2.2.12) to deduce $h = g$. □

The existence of these adjunctions has a useful consequence once we have identified the monads arising from them.

Proposition 2.4.6. *We have the following identities of monads:*

$$\begin{aligned}
(B_{\mathbf{Set}}\ell^1_c, \eta, B_{\mathbf{Set}}\epsilon\ell^1_c) &= (\mathcal{D}, \eta, \mu) \\
(B^{\leq 1}_{\mathbf{Set}}\ell^1_c, \eta, B^{\leq 1}_{\mathbf{Set}}\epsilon\ell^1_c) &= (\mathcal{D}^{\leq 1}, \eta, \mu) \\
(B_{\mathbf{Set}}\ell^1, \eta, B_{\mathbf{Set}}\epsilon\ell^1) &= (\mathcal{D}_\infty, \eta, \mu) \\
(B^{\leq 1}_{\mathbf{Set}}\ell^1, \eta, B^{\leq 1}_{\mathbf{Set}}\epsilon\ell^1) &= (\mathcal{D}^{\leq 1}_\infty, \eta, \mu)
\end{aligned}$$

Proof. That $B_{\mathbf{Set}}\ell^1_c = \mathcal{D}$ and the units are equal, and the analogous statments for the other three monads is immediate for the definitions. We therefore only need to prove that $B_{\mathbf{Set}}\epsilon\ell^1_c = \mu$ and the analogous statements for the other three monads. The argument is virtually the same in all four cases, so we only show that $\mathcal{D}_\infty$ case. The counit arises from the universal property of η in the following manner, where E is a Banach base-norm space:



Therefore, for any $\phi \in \ell^1(B_{\mathbf{Set}}(E))$,

$$\epsilon_E(\phi) = \sum_{x \in B_{\mathbf{Set}}(E)} \phi(x) \cdot x.$$

We can now see that, given $\Phi \in B_{\mathbf{Set}}(\ell^1(B_{\mathbf{Set}}(\ell^1(X)))) = \mathcal{D}^2_\infty(X)$ and $x \in X$ we have

$$\begin{aligned} B_{\mathbf{Set}}(\epsilon_{\ell^1(X)})(\Phi)(x) &= \epsilon_{\ell^1(X)}(\Phi)(x) \\ &= \left(\sum_{\phi \in B_{\mathbf{Set}}(\ell^1(X))} \Phi(\phi) \cdot \phi \right)(x) \\ &= \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \\ &= \mu_X(\Phi)(x). \end{aligned}$$

□

Thus we have a comparison functors

$$B^{\mathcal{D}} : \mathbf{PreBNS} \to \mathcal{EM}(\mathcal{D}) \tag{2.3}$$

$$B^{\mathcal{D}^{\leq 1}} : \mathbf{PreBNS}_{\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1}) \tag{2.4}$$

$$B^{\mathcal{D}_\infty} : \mathbf{BBNS} \to \mathcal{EM}(\mathcal{D}_\infty) \tag{2.5}$$

$$B^{\mathcal{D}^{\leq 1}_\infty} : \mathbf{BBNS}_{\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1}_\infty). \tag{2.6}$$

The monad morphism from $\mathcal{D} \Rightarrow \mathcal{D}_\infty$ induces a functor $\mathcal{EM}(\mathcal{D}_\infty) \to \mathcal{EM}(\mathcal{D})$ (Proposition 2.4.1), so it seems that we have two functors $\mathbf{BBNS} \to \mathcal{EM}(\mathcal{D})$ and $\mathbf{BBNS}_{\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1})$. In fact, they are the same:

Lemma 2.4.7. *The following diagram commutes (strictly)*

$$\begin{array}{ccc} \mathbf{BBNS} & \xrightarrow{B^{\mathcal{D}_\infty}} & \mathcal{EM}(\mathcal{D}_\infty) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbf{PreBNS} & \xrightarrow[B^{\mathcal{D}}]{} & \mathcal{EM}(\mathcal{D}), \end{array}$$

where U is the inclusion functor and V is the functor arising from the monad morphism $\mathcal{D} \Rightarrow \mathcal{D}_\infty$. The analogous diagram for $\mathbf{BBNS}_{\leq 1}$ also commutes.

Proof. It is clear that these functors coincide on morphisms, being restriction to the base, or subbase for the $\mathbf{BBNS}_{\leq 1}$ case. On a Banach base-norm space (E, E_+, τ), the lower left path gives $(B_E, B_{\mathbf{Set}}(\epsilon_E))$, ϵ_E being the counit for the adjunction involving ℓ^1_c. The upper right path gives $(B_E, B_{\mathbf{Set}}(\epsilon_E) \circ \tau_{B_E})$. In Proposition 2.4.6 we saw that ϵ_E for the ℓ^1_c adjunction was the nearly same as the definition for the ℓ^1 adjunction, only restricted to elements of finite support. This is exactly what precomposing with τ_{B_E} does, so $B_{\mathbf{Set}}(\epsilon_E) = B_{\mathbf{Set}}(\epsilon_E) \circ \tau_{B_E}$ and the objects are equal as well. The proof for $\tau^{\leq 1}$ and $\mathbf{BBNS}_{\leq 1}$ is similar. □

We therefore can use the name $B^{\mathcal{D}}$ interchangeably for either functor $\mathbf{BBNS} \to \mathcal{EM}(\mathcal{D})$.

Proposition 2.4.8. *The functors* $B^{\mathcal{D}} : \mathbf{PreBNS} \to \mathcal{EM}(\mathcal{D})$ *and* $B^{\mathcal{D}^{\leq 1}} : \mathbf{PreBNS}_{\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1})$ *are full and faithful, and therefore so are* $B^{\mathcal{D}} : \mathbf{BBNS} \to \mathcal{EM}(\mathcal{D})$ *and* $B^{\mathcal{D}^{\leq 1}} : \mathbf{BBNS}_{\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1})$.

Proof.

- $B^{\mathcal{D}}$ is faithful:

  Let $f, g : E \to F$ in **PreBNS**, with $B^{\mathcal{D}}(f) = B^{\mathcal{D}}(g)$. If $E = 0$, then $f = g$ already, so we reduce to the case that $E \neq 0$. Every element $x \in E$ can be expressed as $\alpha x_+ - \beta x_-$ with $x_+, x_- \in B^{\mathcal{D}}(E)$. Then

  $$\begin{aligned} f(x) &= f(\alpha x_+ - \beta x_-) = \alpha f(x_+) - \beta f(x_-) \\ &= \alpha g(x_+) - \beta g(x_-) = g(\alpha x_+ - \beta x_-) \\ &= g(x). \end{aligned}$$

  The proof for $B^{\mathcal{D}^{\leq 1}}$ is similar.

- $B^{\mathcal{D}}$ is full:

  Consider a map $f : B^{\mathcal{D}}(E) \to B^{\mathcal{D}}(F)$ in $\mathcal{EM}(\mathcal{D})$.

  If $E = 0$, then $B^{\mathcal{D}}(E) = \emptyset$ and f is the unique empty function. Take $g : E \to F$ to be the unique map $0 \to F$. Then $B^{\mathcal{D}}(g) = f$. We therefore reduce to the case that $B^{\mathcal{D}}(E) \neq \emptyset$ and so every $x \in E$ can be expressed as $\alpha x_+ - \beta x_-$, with $x_+, x_- \in B^{\mathcal{D}}(E)$ and $\alpha, \beta \in [0, \infty)$. We then attempt to define $\tilde{f}(\alpha x_+ - \beta x_-) = \alpha f(x_+) - \beta f(x_-)$, and $\tilde{f}(0) = 0$ in the case that $E = 0$. We first prove this defines a function $E \to F$. Suppose $\alpha x_+ - \beta x_- = x = \alpha' x'_+ - \beta' x'_-$. We then have $\alpha x_+ + \beta' x'_- = \alpha' x'_+ + \beta x_-$. Taking the trace of both sides, we get $\alpha + \beta' = \alpha' + \beta$, and we give the name γ to this quantity. If $\gamma = 0$, we have $\alpha = \beta = \alpha' = \beta' = 0$, so $x = 0$, and $0 \cdot f(x_+) - 0 \cdot f(x_-) = 0 = 0 \cdot f(x'_+) - 0 \cdot f(x'_-)$, so $\tilde{f}$ is well-defined in this case.

  Now we can reduce to the case that $\gamma > 0$, so we have the convex combinations $\frac{\alpha}{\gamma} x_+ + \frac{\beta'}{\gamma} x'_-$ and $\frac{\alpha'}{\gamma} x'_+ + \frac{\beta}{\gamma} x_-$. Since f is an $\mathcal{EM}(\mathcal{D})$-morphism, it is affine, so

  $$\begin{aligned} \frac{\alpha}{\gamma} f(x_+) + \frac{\beta'}{\gamma} f(x'_-) = f\left(\frac{\alpha}{\gamma} x_+ + \frac{\beta'}{\gamma} x'_-\right) &= f\left(\frac{\alpha'}{\gamma} x'_+ + \frac{\beta}{\gamma} x_-\right) \\ &= \frac{\alpha'}{\gamma} f(x'_+) + \frac{\beta}{\gamma} f(x_-). \end{aligned}$$

  Multiplying the equation through by γ, we obtain

  $$\alpha f(x_+) + \beta' f(x'_-) = \alpha' f(x'_+) + \beta f(x_-)$$

  and we arrive at the conclusion that

  $$\begin{aligned} \tilde{f}(\alpha x_+ - \beta x_-) &= \alpha f(x_+) - \beta f(x_-) = \alpha' f(x'_+) - \beta' f(x'_-) \\ &= \tilde{f}(\alpha' x'_+ - \beta' x'_-), \end{aligned}$$

which shows that $\tilde{f}$ is well-defined.

We now show that $\tilde{f}$ is linear. Let $x, y \in E$, and decompose them as $x = \alpha x_+ - \beta x_-$ and $y = \gamma y_+ - \delta y_-$. The first case is where $\alpha + \gamma > 0$ and $\beta + \delta > 0$. Then we have

$$\begin{aligned}
\tilde{f}(x+y) &= \tilde{f}\left((\alpha+\gamma)\left(\frac{\alpha}{\alpha+\gamma}x_+ + \frac{\gamma}{\alpha+\gamma}y_+\right)\right. \\
&\quad \left. - (\beta+\delta)\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right)\right) \\
&= (\alpha+\gamma)f\left(\frac{\alpha}{\alpha+\gamma}x_+ + \frac{\gamma}{\alpha+\gamma}y_+\right) \\
&\quad -(\beta+\delta)f\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right) \\
&= \alpha f(x_+) + \gamma f(y_+) - \beta f(x_-) - \delta f(y_-) \\
&= \alpha f(x_+) - \beta f(x_-) + \gamma f(y_+) - \delta f(y_-) \\
&= \tilde{f}(x) + \tilde{f}(y).
\end{aligned}$$

Now, if $\alpha + \gamma = 0$, then $\alpha = \gamma = 0$. So

$$\begin{aligned}
\tilde{f}(x+y) &= \tilde{f}(0 \cdot x_+ - \beta x_- + 0 \cdot y_+ - \delta y_-) \\
&= \tilde{f}\left(0 \cdot \left(\frac{1}{2}x_+ + \frac{1}{2}y_+\right) - (\beta+\delta)\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right)\right) \\
&= 0 \cdot f\left(\frac{1}{2}x_+ + \frac{1}{2}y_+\right) - (\beta+\delta)f\left(\frac{\beta}{\beta+\delta}x_- + \frac{\delta}{\beta+\delta}y_-\right) \\
&= 0 \cdot f(x_+) + 0 \cdot f(y_+) - \beta f(x_-) - \delta f(y_-) \\
&= 0 \cdot f(x_+) - \beta f(x_-) + 0 \cdot f(y_+) - \delta f(y_-) \\
&= \tilde{f}(x) + \tilde{f}(y).
\end{aligned}$$

The case that $\beta + \delta = 0$ is similar. If both are zero, we have $\tilde{f}(0) = 0$, finishing this case. This proves that $\tilde{f}$ is a homomorphism of abelian groups. We now consider multiplication by a real number ξ. There are three cases, $\xi = 0$, $\xi > 0$ and $\xi < 0$. We already have $\xi = 0$, so we concern ourselves only with the other two cases. If $\xi > 0$, we have

$$\begin{aligned}
\tilde{f}(\xi x) &= \tilde{f}(\xi(\alpha x_+ - \beta x_-)) = \tilde{f}(\xi\alpha x_+ - \xi\beta x_-) = \xi\alpha f(x_+) - \xi\beta f(x_-) \\
&= \xi(\alpha f(x_+) - \beta f(x_-)) = \xi(\tilde{f}(\alpha x_+ - \beta x_-)) = \xi\tilde{f}(x).
\end{aligned}$$

If $\xi < 0$, we have

$$\begin{aligned}
\tilde{f}(\xi x) &= \tilde{f}(\xi(\alpha x_+ - \beta x_-)) = \tilde{f}(-\xi\beta x_- - (-\xi\alpha x_+)) \\
&= -\xi\beta f(x_-) - (-\xi\alpha)f(x_+) = \xi\alpha f(x_+) - \xi\beta f(x_-) \\
&= \xi(\alpha f(x_+) - \beta f(x_-)) = \xi(\tilde{f}(\alpha x_+ - \beta x_-)) \\
&= \xi\tilde{f}(x).
\end{aligned}$$

This completes the proof of linearity. If $x \in E_+$, then we can express it as αx_+ for $x_+ \in B^{\mathcal{D}}(E)$, by dividing by its trace or taking $\alpha = 0$ if $x = 0$. We can therefore write it as $\alpha x_+ - \beta x_-$ with $\beta = 0$, taking $x_- = x_+$ if necessary. We have

$$\tilde{f}(x) = \alpha f(x_+) - \beta f(x_-) = \alpha f(x_+),$$

and then because $\alpha \geq 0$ and $f(x_+) \in B^{\mathcal{D}}(F)$ and F_+ is a cone, we have that $\alpha f(x_+) \in F_+$, establishing the positivity of $\tilde{f}$.

For trace-preservation, we observe first that if $E = 0$, the trace $\tau(0) = 0$ and $\sigma(\tilde{f}(0)) = \sigma(0) = 0$, so the trace is preserved. We therefore reduce to the case that $E \neq 0$. For τ we have

$$\tau(x) = \tau(\alpha x_+ - \beta x_-) = \alpha\tau(x_+) - \beta\tau(x_-) = \alpha - \beta.$$

For $\sigma \circ \tilde{f}$ we have

$$\begin{aligned}\sigma(\tilde{f}(x)) &= \sigma(\tilde{f}(\alpha x_+ - \beta x_-)) = \sigma(\alpha f(x_+) - \beta f(x_-)) \\ &= \alpha\sigma(f(x_+)) - \beta\sigma(f(x_-)) = \alpha - \beta.\end{aligned}$$

Since the two are equal, we have trace-preservation as well, and $\tilde{f}$ is a **PreBNS** morphism such that $B^{\mathcal{D}}(\tilde{f}) = f$, as required.

- $B^{\mathcal{D}^{\leq 1}}$ is full: We use the same definition for $\tilde{f}$ as in the $B^{\mathcal{D}}$ case, and then the proofs of well-definedness, linearity and positivity are virtually the same. We therefore only present the proof that $\tilde{f}$ is trace-reducing.

  In the case that $B^{\mathcal{D}^{\leq 1}}(E) = \{0\}$, then $E = 0$ and

  $$(\tau - \sigma \circ \tilde{f})(0) = \tau(0) - \sigma(\tilde{f}(0)) = 0 - 0 = 0$$

  so $\sigma \circ \tilde{f} \leq \tau$. We therefore reduce to the case that $E \neq 0$. If $x \in E_+$¡ then $x = \alpha x_+ - \beta x_-$ with $\beta = 0$, and $x_+, x_- \in B^{\mathcal{D}}(E)$. We have that $\tau(x) = \tau(\alpha x_+) = \alpha$. Then

  $$\begin{aligned}\sigma(\tilde{f}(x)) &= \sigma(\tilde{f}(\alpha x_+ - \beta x_-)) = \sigma(\alpha f(x_+) - \beta f(x_-)) \\ &= \alpha\sigma(f(x_+)) - 0 \leq \alpha = \tau(x),\end{aligned}$$

  which shows that $\tilde{f}$ is trace-reducing. □

The preceding proof also shows the following:

Corollary 2.4.9. *The functor* B : **PreBNS** $\to$ **BConv** *is an equivalence.*

Proof. It is full and faithful by the previous proposition, as affine maps and $\mathcal{EM}(\mathcal{D})$ maps coincide for convex sets. We have already shown that it is essentially surjective in Proposition 2.2.13, so it is definition (iii) of an equivalence (Theorem 0.4.3). □

By combining this corollary with Lemma 2.4.7, we see that if X and Y are $\mathcal{D}_\infty$-algebras isomorphic to bases of pre-base-norm spaces, $\mathcal{EM}(\mathcal{D})(X,Y) = \mathcal{EM}(\mathcal{D}_\infty)(X,Y)$. In fact, we can improve this result somewhat, also extending a result from [106, Theorem 3.6].

Lemma 2.4.10. *Let (X, α_X) and (Y, α_Y) be Eilenberg-Moore algebras of $\mathcal{D}_\infty$, where $Y \cong B_E$ for some pre-base-norm space (E, E_+, τ). Then $\mathcal{EM}(\mathcal{D})(X,Y) = \mathcal{EM}(\mathcal{D}_\infty)(X,Y)$. The analogous result also holds for $\mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\leq 1}_\infty$.*

Proof. As the inclusion map $\mathcal{EM}(\mathcal{D}_\infty)(X,Y) \subseteq \mathcal{EM}(\mathcal{D})(X,Y)$ is natural, we can reduce to the case that $Y = B_E$. All we need to show is that if $a : X \to Y$ is a map of $\mathcal{D}$-algebras, it is a map of $\mathcal{D}_\infty$-algebras, *i.e.* that

$$\begin{array}{ccc} \mathcal{D}_\infty(X) & \xrightarrow{\mathcal{D}_\infty(a)} & \mathcal{D}_\infty(Y) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow[a]{} & Y \end{array}$$

commutes. By Proposition 2.4.8, $a \circ \alpha_X$ and $\alpha_Y \circ \mathcal{D}_\infty(a)$ extend to trace-preserving maps $b, c : \ell^1(X) \to E$, which agree on $\ell^1_c(X)$ because a is a map of $\mathcal{D}$-algebras. By Proposition 2.2.12, b and c are bounded, and therefore continuous, and by Lemma 2.4.2 the set $\ell^1_c(X)$, on which they agree, is dense, $b = c$. Therefore $a \circ \alpha_X = \alpha_Y \circ \mathcal{D}_\infty(a)$, as required.

The proof for $\mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\leq 1}_\infty$ is similar, using the $\mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\leq 1}_\infty$ parts of the previously mentioned results. $\square$

We can define $(2, \alpha_2)$ as $2 = \{0, \infty\}$, $\alpha_2(\eta_2(0)) = 0$ and $\alpha_2(\phi) = \infty$ otherwise, and a map $f : \mathcal{D}_\infty(\mathbb{N}) \to 2$ such that $f(\phi) = 0$ if ϕ has finite support and ∞ if it has infinite support. It is left as an exercise to the reader to show that 2 is an Eilenberg-Moore algebra of $\mathcal{D}_\infty$ and f a map in $\mathcal{EM}(\mathcal{D})$. If we define $\Phi \in \mathcal{D}^2_\infty(\mathbb{N})$ as $\Phi(\eta_\mathbb{N}(i)) = 2^{-i}$, we have $\alpha_2(\mathcal{D}_\infty(f)(\Phi)) = 0$ because each $\eta_\mathbb{N}(i)$ has finite support, but $f(\mu_\mathbb{N}(\Phi)) = \infty$. This shows that the forgetful functor $V : \mathcal{EM}(\mathcal{D}_\infty) \to \mathcal{EM}(\mathcal{D})$ is not full.

The following was first proven in [106, Lemma 3.2] in the setting of superconvex sets.

Proposition 2.4.11. *Let (E, E_+, τ) be a pre-base-norm space. If B_E, or equivalently $B^{\leq 1}_E$, is σ-convex, then E is a Banach space in the base norm.*

Proof. We first show that B_E is σ-convex iff $B^{\leq 1}_E$ is. If $B^{\leq 1}_E$ is σ-convex, then any σ-convex combination $\sum_{i=1}^\infty \alpha_i x_i$ where $x_i \in B_E$ is also a σ-convex combination in $B^{\leq 1}_E$, so the sum converges to some $x \in B^{\leq 1}_E$. By Lemma 2.2.4

$$\tau\left(\sum_{i=1}^\infty \alpha_i x_i\right) = \sum_{i=1}^\infty \tau(\alpha_i x_i) = \sum_{i=1}^\infty \alpha_i = 1,$$

so $x \in B_E$, so B_E is σ-convex.

In the other direction, suppose B_E is σ-convex, and let $\sum_{i=1}^{\infty} \alpha_i x_i$ be a σ-convex combination in $B_E^{\leq 1}$. If $x_i = 0$ for all i or all but finitely many i, then $\sum_{i=1}^{\infty} \alpha_i x_i$ is actually a finite convex combination, and we are done. Therefore we reduce to the case that $x_i \neq 0$ for infinitely many i. Define (y_i) to be the subsequence of non-zero terms, and (β_i) to be the corresponding elements in (α_i), and we have that $\sum_{i=1}^{\infty} \beta_i y_i$ converges iff $\sum_{i=1}^{\infty} \alpha_i x_i$ does and if they do they have the same limit. Then define $z_i = \frac{y_i}{\tau(y_i)}$, which avoids dividing by zero by Lemma 2.2.2, and gives a sequence in B_E. As $\sum_{i=1}^{\infty} \alpha_i$ converges, we have that $\sum_{i=1}^{\infty} \beta_i$ converges absolutely, though perhaps to some value less than 1. We define $\beta = \sum_{i=1}^{\infty} \beta_i$, and $\gamma_i = \frac{\beta_i \tau(y_i)}{\beta}$. As the sequence $\frac{\tau(y_i)}{\beta}$ is bounded, we have

$$\sum_{i=1}^{\infty} \gamma_i = \sum_{i=1}^{\infty} \beta_i \cdot \frac{\tau(y_i)}{\beta}$$

converges, and by continuity of scalar multiplication and the definition of β its value is 1. Therefore $\sum_{i=1}^{\infty} \gamma_i z_i$ is a σ-convex combination in B_E, and we name the limit of it z. We then have

$$\beta z = \beta \sum_{i=1}^{\infty} \gamma_i z_i = \sum_{i=1}^{\infty} \beta \cdot \frac{\beta_i \tau(y_i)}{\beta} \cdot \frac{y_i}{\tau(y_i)} = \sum_{i=1}^{\infty} \beta_i y_i.$$

Therefore $\sum_{i=1}^{\infty} \alpha_i x_i$ converges to βz, so $B_E^{\leq 1}$ is σ-convex.

By Corollary 2.2.5, $\mathrm{Ball}(\|\text{-}\|) \cap E_+ = B_E^{\leq 1}$, so if $B_E^{\leq 1}$ is σ-convex, we may apply Lemma 2.2.15 to conclude that $E = E_+ - E_+$ is complete in $\|\text{-}\|_{\mathrm{co}(-B^{\leq 1} \cup B^{\leq 1})}$, which is the base norm by Lemma 2.2.11. □

Lemma 2.4.12.

(i) Let (X, α_X) be a $\mathcal{D}$-algebra isomorphic to B_E for some pre-base-norm space (E, E_+, τ). If $\beta_X, \gamma_X : \mathcal{D}_\infty(X) \to X$ are $\mathcal{D}_\infty$-algebra structures agreeing on $\mathcal{D}(X)$ with α_X, then $\beta_X = \gamma_X$, and B_E is σ-convex for the base norm.

(ii) If (E, X) is an object of **BConv**, *$\mathcal{T}$ is the topology on E, and X is σ-convex in $\mathcal{T}$, then σ-convex combinations extend the $\mathcal{D}$-algebra structure of X to a $\mathcal{D}_\infty$-algebra structure, so X is σ-convex in the base norm of the base-norm space constructed in Proposition 2.2.13.*

Proof.

(i) Let $i : (X, \alpha_X) \to B_E$ be an $\mathcal{EM}(\mathcal{D})$ isomorphism, and $\beta_X, \gamma_X : \mathcal{D}_\infty(X) \to X$ $\mathcal{EM}(\mathcal{D}_\infty)$-structures on X extending α_X. Then β_X and γ_X are $\mathcal{EM}(\mathcal{D}_\infty)$-maps from $\mathcal{D}_\infty(X) \to X$ by the definition of an Eilenberg-Moore algebra. Therefore they are also $\mathcal{EM}(\mathcal{D})$-maps (Proposition 2.4.1), and so $\imath \circ \beta_X$ and $i \circ \gamma_X$ are $\mathcal{EM}(\mathcal{D})$-maps between bases of pre-base-norm spaces, and therefore extend to trace-preserving maps $f, g : \ell^1(X) \to E$ by Proposition 2.4.8. The maps f, g agree when restricted to $\mathcal{D}(X)$, so agree on $\ell^1_c(X)$ by linearity. As they are continuous (Proposition 2.2.12) and $\ell^1_c(X)$ is dense (Lemma 2.4.2), $f = g$. Therefore $i \circ \beta_X = i \circ \gamma_X$, so $\beta_X = \gamma_X$.

We now show that B_E is σ-convex. Let $\sum_{i=1}^{\infty} \alpha_i y_i$ be a σ-convex combination of elements of B_E. Without loss of generality, take $y_i \neq y_j$ for $i \neq j$ and $\alpha_i \neq 0$ for all i. Let $x_j = i^{-1}(y_j)$, which is unique because i is a bijection. Define $\phi \in \mathcal{D}_\infty(X)$ as

$$\begin{aligned} \phi(x_i) &= \alpha_i && \text{if } 1 \leq i \leq n \\ \phi(x) &= 0 && \text{otherwise,} \end{aligned}$$

which is in $\mathcal{D}^1_\infty(X)$ because α_i is the coefficients of an absolutely convex combination, and define $\phi_n \in \ell^1_c(X)$ as

$$\begin{aligned} \phi_n(x_i) &= \alpha_i && \text{if } 1 \leq i \leq n \\ \phi_n(x) &= 0 && \text{otherwise.} \end{aligned}$$

We have

$$\|\phi - \phi_n\| = \sum_{i=n+1}^{\infty} |\phi(x_i)| = \sum_{i=n+1}^{\infty} \alpha_i$$

so $\phi_n \to \phi$ in the ℓ^1 norm. We know

$$f|_{\mathcal{D}(X)} = i \circ \alpha_X = B(\epsilon_E) \circ \mathcal{D}(i)$$

because i is an $\mathcal{EM}(\mathcal{D})$-morphism and the Eilenberg-Moore structure on B_E comes from a comparison functor. We can define $\beta_n = \sum_{i=1}^{n} \alpha_i$ and $\psi_n = \frac{\phi_n}{\beta_n}$, it is then an element of $\mathcal{D}(X)$, so

$$f(\psi_n) = B(E)(\mathcal{D}(i)(\psi_n)) = \epsilon_E(\psi_n \circ i^{-1}).$$

By the definition of ϵ_E from Proposition 2.4.6, this is

$$\sum_{y \in B_E} (\psi_n \circ i^{-1})(y) \cdot y = \sum_{j=1}^{n} \psi_n(x_j) \cdot y_j = \frac{\sum_{j=1}^{n} \phi_n(x_j) y_j}{\beta_n} = \frac{\sum_{j=1}^{n} \alpha_j y_j}{\beta_n}.$$

By linearity of f, we can cancel the β_n and get $f(\phi_n) = \sum_{j=1}^{n} \alpha_j y_j$. As f is continuous (Proposition 2.2.12), $f(\phi_n) \to f(\phi)$, so

$$\sum_{j=1}^{n} \alpha_j y_j \to f(\phi),$$

so $\sum_{j=1}^{\infty} \alpha_j y_j$ converges to an element of B_E. Therefore B_E is σ-convex in the base norm.

(ii) We define a $\mathcal{D}_\infty$-algebra structure $\beta_X : \mathcal{D}_\infty(X) \to X$ as follows, where $\phi \in \mathcal{D}_\infty(X)$:

$$\beta_X(\phi) = \sum_{x \in X} \phi(x) \cdot x.$$

By Lemma 0.1.9, the sum defining $\beta_X(\phi)$ is a σ-convex combination, so defines an element of X. This definition extends the $\mathcal{D}$-algebra structure α_X defined by Propositions 2.2.13 and 2.4.6. We still need to show that it makes (X, β_X) an Eilenberg-Moore algebra. We have $\beta_X \circ \eta_X = \mathrm{id}_X$ because the range of η_X lies inside $\mathcal{D}(X)$, and we already have $\alpha_X \circ \eta_X = \mathrm{id}_X$. Therefore we need to show

$$\begin{array}{ccc} \mathcal{D}^2_\infty(X) & \xrightarrow{\mathcal{D}_\infty(\beta_X)} & \mathcal{D}_\infty(X) \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \beta_X} \\ \mathcal{D}_\infty(X) & \xrightarrow[\beta_X]{} & X. \end{array}$$

Let $\Phi \in \mathcal{D}^2_\infty(X)$. The lower right path gives

$$\begin{aligned} \beta_X(\mu_X(\Phi)) &= \sum_{x \in X} \mu_X(\Phi)(x) \cdot x \\ &= \sum_{x \in X} \left(\sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \right) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \cdot x. \end{aligned}$$

The upper right path gives

$$\begin{aligned} \beta_X(\mathcal{D}_\infty(\beta_X)(\Phi)) &= \sum_{x \in X} \mathcal{D}_\infty(\beta_X)(\Phi)(x) \cdot x \\ &= \sum_{x \in X} \left(\sum_{\phi \in \beta_X^{-1}(x)} \Phi(\phi) \right) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \beta_X^{-1}(x)} \Phi(\phi) \cdot x. \end{aligned}$$

For each $\phi \in \mathcal{D}_\infty(X)$, there is a unique x such that $\beta_X(\phi) = x$, and all values of x occur, so we can rewrite the above expression as

$$\begin{aligned} \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \beta_X(\phi) &= \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \left(\sum_{x \in X} \phi(x) \cdot x \right) \\ &= \sum_{\phi \in \mathcal{D}_\infty(X)} \sum_{x \in X} \Phi(\phi) \cdot \phi(x) \cdot x \\ &= \sum_{x \in X} \sum_{\phi \in \mathcal{D}_\infty(X)} \Phi(\phi) \cdot \phi(x) \cdot x, \end{aligned}$$

by the absolute convergence of the sums. Therefore the diagram commutes.

We can then apply the previous part to conclude that the base of the corresponding base-norm space is σ-convex in the base norm. □

We define **CBConv** to be the full subcategory of **BConv** on objects (E, X) where X is sequentially complete in the subspace uniformity of E. If (E, E_+, τ) is a Banach base-norm space, then B_E is a closed subspace of the complete space E, so (E, B_E) is an object of **CBConv**, and the functor $B : \mathbf{PreBNS} \to \mathbf{BConv}$ restricts to $B : \mathbf{BBNS} \to \mathbf{CBConv}$.

Proposition 2.4.13. *The functor* $B : \mathbf{BBNS} \to \mathbf{CBConv}$ *is an equivalence of categories.*

Proof. By Proposition 2.4.8 we have that B is full and faithful because $\mathcal{EM}(\mathcal{D})$ maps are the same as **CBConv** maps. By Proposition 2.2.13, we have that for any object (E, X) in **CBConv**, where $\mathcal{T}$ is the topology of E, there is a pre-base-norm space (F, F_+, τ), with a locally convex topology $\mathcal{S}$ on F such that $(E, X) \cong (F, B_F)$ in **BConv** and this isomorphism is a uniform isomorphism with respect to the uniformities on X and B_F induced by $\mathcal{T}$ and $\mathcal{S}$ respectively. Since X is sequentially complete, by 0.1.19 it is σ-convex, so by Lemma 2.4.12 B_F is σ-convex in the base norm. By Proposition 2.4.11, F is a Banach space in the base norm. All that is left to show $(F, F_+\tau)$ is a Banach base-norm space is to show that F_+ is closed in the base norm. As X is sequentially complete in $\mathcal{T}$, we have that B_F is sequentially complete in $\mathcal{S}$, and therefore sequentially closed. By Lemma 2.2.14, F_+ is therefore sequentially closed in $(F, \mathcal{S})$. As the base-norm topology is finer than $\mathcal{S}$, we have that F_+ is sequentially closed, and therefore closed, in the base norm. □

If we defined **CBConvBan** to be closed (and therefore complete) bounded convex subsets of Banach spaces, the forgetful functor $\mathbf{CBConvBan} \to \mathbf{CBConv}$ is an equivalence by the previous proposition, because the image of **BBNS** under B lies inside **CBConvBan**. In other words, every sequentially complete bounded convex subset of a locally convex space is embeddable as a closed subset of a Banach space. It is not the case that every object of **CCL** can be embedded in a Banach space (with the norm topology, at least) because some objects, such as $[0, 1]^X$ for uncountable X, are not first-countable, and therefore not metrizable.

2.4.2 Bounded Affine Functions and the Dual Space

If (X, α_X) is an Eilenberg-Moore algebra of $\mathcal{D}$, we can define the real-valued bounded affine functions $\mathrm{BAff}(X, \alpha_X)$, and for an Eilenberg-Moore algebra of $\mathcal{D}_\infty$, we can define the bounded σ-affine functions $\mathrm{BAff}_\infty(X, \alpha_X)$, where we require that infinite convex combinations are preserved. In more detail

$$\begin{aligned}
&\mathrm{BAff}(X, \alpha_X) \\
&= \left\{ a : X \to \mathbb{R} \,\middle|\, a \in \ell^\infty(X) \text{ and } \forall \phi \in \mathcal{D}(X). a(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot a(x) \right\} \\
&\mathrm{BAff}_\infty(X, \alpha_X) \\
&= \left\{ a : X \to \mathbb{R} \,\middle|\, a \in \ell^\infty(X) \text{ and } \forall \phi \in \mathcal{D}_\infty(X). a(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot a(x) \right\},
\end{aligned}$$

where the sum in the definition of $\mathrm{BAff}_\infty(X, \alpha_X)$ will always converge by Lemmas 0.1.9 and 0.1.19. We will sometimes use X to refer to (X, α_X) when there is no possibility of confusion, and therefore $\mathrm{BAff}(X, \alpha)$ will sometimes be written as $\mathrm{BAff}(X)$. We will soon see that BAff_∞ and BAff are actually the same. To do this, we need a definition. For each $\alpha \in \mathbb{R}_{>0}$ we can define $[-\alpha, \alpha] \subseteq \mathbb{R}$. As it is a complete subset of $\mathbb{R}$, $(\mathbb{R}, [-\alpha, \alpha])$ is an object of **CBConv**, so it admits a $\mathcal{D}_\infty$-algebra structure. We can construct a $\mathcal{D}$-algebra isomorphism $i_\alpha : [-\alpha, \alpha] \to B^{\mathcal{D}_\infty}(\mathbb{R}^2)$ as follows

$$i_\alpha(x) = \left(\frac{\alpha - x}{2\alpha}, \frac{\alpha + x}{2\alpha} \right). \tag{2.7}$$

This map is a $\mathcal{D}$-algebra homomorphism because it is affine, and it is therefore a $\mathcal{D}_\infty$-algebra homomorphism by Lemma 2.4.10. As it is an isomorphism, it is also a $\mathcal{D}_\infty$-algebra isomorphism.

Lemma 2.4.14. *For all $\mathcal{D}_\infty$-algebras (X, α_X), $\mathrm{BAff}_\infty(X) = \mathrm{BAff}(X)$.*

Proof. By definition, $\mathrm{BAff}_\infty(X) \subseteq \mathrm{BAff}(X)$. For the opposite inclusion, let $a \in \mathrm{BAff}(X)$, and then there is an $\alpha \in \mathbb{R}_{>0}$ such that $a(X) \subseteq [-\alpha, \alpha]$. So a can be considered to be a $\mathcal{D}$-algebra morphism $a : X \to [-\alpha, \alpha]$. Therefore $i_\alpha \circ a : X \to B(\mathbb{R}^2)$ is a $\mathcal{D}$-algebra map, so by Lemma 2.4.10 it is an $\mathcal{EM}(\mathcal{D}_\infty)$-algebra map. Therefore $a = i^{-1} \circ i \circ a$ is a $\mathcal{D}_\infty$-algebra homomorphism, and therefore an element of $\mathrm{BAff}_\infty(X)$ as a map $X \to \mathbb{R}$. □

We can give $\mathrm{BAff}(X, \alpha)$ the order-unit space structure they should have as a subspace of $\ell^\infty(X)$, *i.e.* the vector space operations are pointwise, define $a : X \to \mathbb{R}$ to be positive if $a(x) \geq 0$ for all $x \in X$, and define the unit to be the map such that $u(x) = 1$ for all $x \in X$.

Proposition 2.4.15. *The preceding definitions make $\mathrm{BAff}(X, \alpha)$ into a Banach order-unit space.*

Proof.

- $\mathrm{BAff}(X)$ is a subspace of ℓ^∞: We first show closure under addition. Let $a, b \in \mathrm{BAff}(X)$. We know that $a + b \in \ell^\infty(X)$, so we only want to show that for all $\phi \in \mathcal{D}(X)$ that

$$(a + b)(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot (a + b)(x).$$

So let $\phi \in \mathcal{D}(X)$. Then

$$\begin{aligned}(a+b)(\alpha_X(\phi)) = a(\alpha_X(\phi)) + b(\alpha_X(\phi)) &= \sum_{x\in X}\phi(x)a(x) + \sum_{x\in X}\phi(x)b(x) \\ &= \sum_{x\in X}(\phi(x)a(x) + \phi(x)b(x)).\end{aligned}$$

The last step is because the sum is finite. We then have that this is equal to

$$= \sum_{x\in X}\phi(x)\cdot(a+b)(x),$$

as required.

To show closure under multiplication, let $\beta \in \mathbb{R}$ and $a \in \mathrm{BAff}(X)$. Then $\beta a \in \ell^\infty(X)$, and given $\phi \in \mathcal{D}(X)$

$$(\beta a)(\alpha_X(\phi)) = \beta a(\alpha_X(\phi)) = \beta\left(\sum_{x\in X}\phi(x)\cdot a(x)\right) = \sum_{x\in X}\phi(x)\cdot(\beta a)(x).$$

- The positive cone is a cone: This follows directly from the fact that it is the restriction of $\ell^\infty(X)$'s positive cone.

- The unit is an element of $\mathrm{BAff}(X)$:

  We already know it is bounded, so we only need to show it is affine. So let $\phi \in \mathcal{D}(X)$:

$$u(\alpha_X(\phi)) = 1 = \sum_{x\in X}\phi(x) = \sum_{x\in X}\phi(x)\cdot u(x).$$

- 1 is a strong archimedean unit: This follows directly from it being a strong archimedean unit in $\ell^\infty(X)$.

- $\mathrm{BAff}(X)$ is complete in the norm defined by $[-u, u]$:

  As they are subspaces of $\ell^\infty(X)$ and $\ell^\infty(X)$'s norm is defined by $[-u, u]$, it suffices to show that $\mathrm{BAff}(X)$ is a closed subspace of $\ell^\infty(X)$. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{BAff}(X)$ uniformly converging to $a \in \ell^\infty(X)$. If $\phi \in \mathcal{D}(X)$ then

$$\begin{aligned}a(\alpha_X(\phi)) = \lim_{i\to\infty} a_i(\alpha_X(\phi)) &= \lim_{i\to\infty}\sum_{x\in X}\phi(x)\cdot a_i(x) \\ &= \sum_{x\in X}\phi(x)\left(\lim_{i\to\infty} a_i\right)(x) = \sum_{x\in X}\phi(x)a(x).\end{aligned}$$

  so a is in $\mathrm{BAff}(X)$. □

If $f : (X, \alpha_X) \to (Y, \alpha_Y)$ is an $\mathcal{EM}(\mathcal{D})$ morphism, we can define

$$\begin{aligned}\mathrm{BAff}(f) : \mathrm{BAff}(Y) \to \mathrm{BAff}(X)\\ \mathrm{BAff}(f)(b) = b \circ f,\end{aligned}$$

where $b \in \mathrm{BAff}(Y)$.

Theorem 2.4.16. *These definitions make a functor*

$$\mathrm{BAff} : \mathcal{EM}(\mathcal{D}) \to \mathbf{BOUS}^{\mathrm{op}}$$

Proof. Let $f : (X, \alpha_X) \to (Y, \alpha_Y)$ be an $\mathcal{EM}(\mathcal{D})$ morphism, and b an element of $\mathrm{BAff}(Y)$. First we show that $b \circ f \in \mathrm{BAff}(X)$. We know that b is bounded, so there must exist an $\alpha \in \mathbb{R}_{\geq 0}$ such that $\forall y \in Y.|b(y)| \leq \alpha$. Since for all $x \in X$, $f(x) \in Y$, we have that $\forall x \in X.|b(f(x))| \leq \alpha$, so $b \circ f$ is bounded.

For the affineness, let $\phi \in \mathcal{D}(X)$. We want to show that

$$\mathrm{BAff}(f)(b)(\alpha_X(\phi)) = \sum_{x \in X} \phi(x) \cdot \mathrm{BAff}(f)(b)(x). \tag{2.8}$$

Starting with the left hand side, we have

$$\begin{aligned}\mathrm{BAff}(f)(b)(\alpha_X(\phi)) &= b(f(\alpha_X(\phi)))\\ &= b(\alpha_Y(\mathcal{D}(f)(\phi))) && f \text{ an } \mathcal{EM}(\mathcal{D}) \text{ map}\\ &= \sum_{y \in Y} \mathcal{D}(f)(\phi)(y) \cdot b(y) && b \in \mathrm{BAff}(Y)\\ &= \sum_{y \in Y} b(y) \cdot \left(\sum_{x \in f^{-1}(y)} \phi(x) \right),\end{aligned}$$

by the definition of $\mathcal{D}(f)$. Now, if we look at the right hand side of (2.8), we get

$$\begin{aligned}\sum_{x \in X} \phi(x) \cdot \mathrm{BAff}(f)(b)(x) &= \sum_{x \in X} \phi(x) \cdot b(f(x))\\ &= \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) \cdot b(f(x)) && \text{finite sum}\\ &= \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \phi(x) \cdot b(y) && f(x) = y\\ &= \sum_{y \in Y} b(y) \left(\sum_{x \in f^{-1}(y)} \phi(x) \right),\end{aligned}$$

so we have proved (2.8).

Now we must show that $\mathrm{BAff}(f)$ is a linear positive unital map. The linearity follows from the pointwiseness of the operations defined on $\mathrm{BAff}(X)$. To show $\mathrm{BAff}(f)$ is positive, let $b \in \mathrm{BAff}(Y)_+$, *i.e.* $b(y) \geq 0$ for all $y \in Y$. Then

$$\mathrm{BAff}(f)(b)(x) = b(f(x)) \geq 0$$

for all $x \in X$, so $\mathrm{BAff}(f)(b) \in \mathrm{BAff}(X)$.

Since

$$\mathrm{BAff}(f)(u)(x) = u(f(x)) = 1$$

we have $\mathrm{BAff}(f)(u) = u$, so $\mathrm{BAff}(f)$ is unital.

Finally, BAff preserves identity maps because $a \circ \mathrm{id}_X = a$ and preserves composition by associativity of composition. □

If we have a bounded linear functional $\phi : E \to \mathbb{R}$ where E is a pre-base-norm space, then $\phi|_{B^{\mathcal{D}}(E)}$ is a bounded affine function on $B^{\mathcal{D}}(E)$. This defines a restriction map $\rho_E : E^* \to \mathrm{BAff}(B^{\mathcal{D}}(E))$. In addition to its Banach space structure as a dual space, we can define a positive cone

$$E^*_+ = \{\phi \in E^* | \forall x \in E_+ . \phi(x) \geq 0\},$$

which is a cone, and not merely a wedge, by Lemma 0.3.8, and we can define a unit element τ, as by Lemma 2.2.4 $\tau \in E^*$ and as it is positive it defines an element of E^*_+.

We can then prove a slight generalization of [6, Proposition 1.11].

Proposition 2.4.17. *The map $\rho_E : E^* \to \mathrm{BAff}(B^{\mathcal{D}}(E))$ is a linear isomorphism preserving the positive cone and unit both ways. Therefore E^* is a Banach order-unit space for any pre-base-norm space E, and the closed unit ball as a dual space is exactly the interval $[-\tau, \tau]$. In the case that E has σ-convex base, such as when $E \in$ **BBNS**, ρ_E defines an isomorphism $E^* \to \mathrm{BAff}(B^{\mathcal{D}_\infty}(E))$.*

Proof. Let $a \in E^*$. We first show that $a|_{B^{\mathcal{D}}(E)} \in \mathrm{BAff}(B^{\mathcal{D}}(E))$. As $B^{\mathcal{D}}(E) \subseteq \mathrm{Ball}(E)$ (Lemma 0.1.6), a is bounded on $B^{\mathcal{D}}(E)$. If $\phi \in \mathcal{D}(B^{\mathcal{D}}(E))$, then

$$\begin{aligned} a(\alpha_{B^{\mathcal{D}}(E)}(\phi)) = a(B^{\mathcal{D}}(\epsilon_E)(\phi)) &= a\left(\sum_{x \in B^{\mathcal{D}}(E)} \phi(x) \cdot x\right) \\ &= \sum_{x \in B^{\mathcal{D}}(E)} \phi(x) \cdot a(x), \end{aligned}$$

so $\rho_E(a) \in \mathrm{BAff}(E)$. The map ρ_E is linear because the addition is pointwise. It is injective because if $\rho_E(a) = \rho_E(b)$ then a and b agree on $B^{\mathcal{D}}(E)$, and as E is the span of $B^{\mathcal{D}}(E)$, $a = b$.

To show that ρ_E is surjective, let $a \in \mathrm{BAff}(B^{\mathcal{D}}(E))$. Since it is bounded, there exists an $\alpha \in \mathbb{R}_{\geq 0}$ such that $|a(x)| \leq \alpha$ for all $x \in B^{\mathcal{D}}(E)$. Reusing the affine isomorphism i_α from (2.7), we have $i_\alpha \circ a : B^{\mathcal{D}}(E) \to B^{\mathcal{D}}(\mathbb{R}^2)$, and so by Proposition 2.4.8 it extends to a trace-preserving map $\overline{i_\alpha \circ a} : E \to \mathbb{R}^2$. We can define a linear map

$$\begin{aligned} p_\alpha &: \mathbb{R}^2 \to \mathbb{R} \\ p_\alpha(x, y) &= -\alpha x + \alpha y, \end{aligned}$$

and as this is a map between finite dimensional spaces, this is bounded. Therefore $p_\alpha \circ \overline{i_\alpha \circ a} \in E^*$. Now, let $x \in B^{\mathcal{D}}(E)$ in the following

$$\begin{aligned}\rho_E(p_\alpha \circ \overline{i_\alpha \circ a})(x) &= p_\alpha(i_\alpha(a(x))) = p_\alpha\left(\frac{\alpha - a(x)}{2\alpha}, \frac{a(x)+\alpha}{2\alpha}\right) \\ &= \frac{a(x)-\alpha}{2} + \frac{a(x)+\alpha}{2} = a(x).\end{aligned}$$

This proves that ρ_E is a linear bijection.

We chose τ to be the unit element of E^*. Now, for any element $x \in B^{\mathcal{D}}(E)$, $\tau(x) = 1$, so $\rho_E(\tau) = u$. Now, if $a \in E^*_+$, then since $B^{\mathcal{D}}(E) \subseteq E_+$, we have that $\rho_E(a)(x) \geq 0$ for all $x \in B^{\mathcal{D}}(E)$, and so $\rho_E(a) \in \mathrm{BAff}(B^{\mathcal{D}}(E))_+$. If, on the other hand, $\rho_E(a) \in \mathrm{BAff}(B^{\mathcal{D}}(E))$, then, as each $x \in E_+$ can be expressed as $\alpha x'$ for $x' \in B^{\mathcal{D}}(E)$ and $\alpha \in \mathbb{R}_{\geq 0}$, so

$$a(x) = a(\alpha x') = \alpha a(x') \geq 0$$

and $a \in E^*_+$. Therefore E^* is a Banach order-unit space and ρ_E an isomorphism in **BOUS** between E^* and $\mathrm{BAff}(B^{\mathcal{D}}(E))$.

All that is left to prove is that the usual unit ball of E^* as a dual space coincides with $[-\tau, \tau]$. Suppose $a \in \mathrm{Ball}(E^*)$. Then for all $x \in \mathrm{Ball}(E)$, we have $|a(x)| \leq 1$, or $1 \leq a(x) \leq 1$. By Lemma 0.1.6, $B^{\mathcal{D}}(E) \subseteq \mathrm{Ball}(E)$, so for all $x \in B^{\mathcal{D}}(E)$ we have $-u(x) = -1 \leq \rho_E(a)(x) \leq 1 = u(x)$. As ρ_E is a poset isomorphism and $\rho_E(\tau) = u$, we have shown $-\tau \leq a \leq \tau$.

For the other direction, suppose that $-\tau \leq a \leq \tau$. Suppose for a contradiction that there exists some $x \in \mathrm{Ball}(E)$ such that $|a(x)| > 1$, taking $\alpha = |a(x)|$. So

$$\left\|\frac{2}{1+\alpha}x\right\| = \frac{2}{1+\alpha}\|x\| = \frac{2}{1+\alpha} < 1.$$

By Lemma 0.1.6, $\frac{2}{1+\alpha}x \in \mathrm{absco}(B^{\mathcal{D}}(E))$. If $B^{\mathcal{D}}(E) = \emptyset$, then $a(x) = 0$ for all $x \in E$ so we have a contradiction. Therefore we reduce to the case that $B^{\mathcal{D}}(E) \neq \emptyset$, and therefore $\frac{2}{1+\alpha}x = \beta x_+ + (1-\beta)x_-$, with $x_+, x_- \in B^{\mathcal{D}}(E)$ and $\beta \in [0,1]$. By the assumption on a, we have $-1 \leq a(x_\pm) \leq 1$, or $|a(x_\pm)| \leq 1$. Therefore

$$\begin{aligned}\left|a\left(\frac{2}{1+\alpha}x\right)\right| &= |a(\beta x_+ + (1-\beta)x_-)| \leq \beta|a(x_+)| + (1-\beta)|a(x_-)| \\ &\leq \beta + 1 - \beta = 1.\end{aligned}$$

By linearity of a, this implies

$$|a(x)| \leq \frac{1+\alpha}{2},$$

but this contradicts $|a(x)| = \alpha > 1$.

Finally, the statement for $\mathrm{BAff}(B^{\mathcal{D}_\infty}(E))$ follows by Lemma 2.4.14. □

Given a trace-preserving map $f : E \to F$, we can define $f^* : F^* \to E^*$ as $f^*(b) = b \circ f$.

Theorem 2.4.18. *This definition makes* -^* *a functor* $\mathbf{PreBNS} \to \mathbf{BOUS}^{\mathrm{op}}$*, and* ρ *is a natural isomorphism* $\text{-}^* \Rightarrow \mathrm{BAff} \circ B^{\mathcal{D}}$*, and also from* $\text{-}^* \Rightarrow \mathrm{BAff} \circ B^{\mathcal{D}_\infty}$*.*

Proof. We do the proof only for $\mathrm{BAff} \circ B^{\mathcal{D}}$, using Lemma 2.4.14 for $B^{\mathcal{D}_\infty}$.

If we show that the naturality diagram commutes, and then that this implies that the definition of -^* on maps is a functor and ρ is a natural transformation. The diagram in question, for $f \in \mathbf{PreBNS}(E, F)$, is

$$\begin{array}{ccc} F^* & \xrightarrow{\rho_F} & \mathrm{BAff}(B^{\mathcal{D}}(F)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \mathrm{BAff}(B^{\mathcal{D}}(f))} \\ E^* & \xrightarrow[\rho_E]{} & \mathrm{BAff}(B^{\mathcal{D}}(E)). \end{array}$$

To show that this commutes, let $b \in F^*$ and $x \in B^{\mathcal{D}}(E)$. Then

$$\begin{aligned} \rho_E(f^*(b))(x) &= f^*(b)(x) = b(f(x)) = \rho_F(b)(B^{\mathcal{D}}(f)(x)) \\ &= \mathrm{BAff}(B^{\mathcal{D}}(f))(\rho_F(b))(x). \end{aligned}$$

The commutativity of the diagram implies that for any $f \in \mathbf{PreBNS}(E, F)$, $\rho_E^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(f)) \circ \rho_F = f^*$. Since each of the maps composing to give f^* is linear, positive and unital, this proves that f^* is. We can therefore show

$$\begin{aligned} \mathrm{id}_E^* &= \rho_E^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(\mathrm{id}_E)) \circ \rho_E \\ &= \rho_E^{-1} \circ \rho_E && \mathrm{BAff},\ B^{\mathcal{D}} \text{ functors} \\ &= \mathrm{id}_{E^*}. \end{aligned}$$

In the case that $f \in \mathbf{PreBNS}(E, F)$ and $g \in \mathbf{PreBNS}(F, G)$, we have

$$\begin{aligned} (g \circ f)^* &= \rho_E^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(g \circ f)) \circ \rho_G \\ &= \rho_E^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(f)) \circ \mathrm{BAff}(B^{\mathcal{D}}(g)) \circ \rho_G \\ &= \rho_E^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(f)) \circ \rho_F \circ \rho_F^{-1} \circ \mathrm{BAff}(B^{\mathcal{D}}(g)) \circ \rho_G \\ &= f^* \circ g^*, \end{aligned}$$

showing that -^* is a contravariant functor. The diagram we started with then shows that ρ_E is natural. □

2.5 Dualities with Order-Unit Spaces

In this section, we will prove certain categorical duality results for base-norm and order-unit spaces. We start with a dual adjunction between **PreBNS** and **OUS**, related to that between $\mathcal{EM}(\mathcal{D})$ and **EA** in [56]. We then see how this adjunction restricts to an equivalence. As the equivalence derived from it is not entirely satisfactory, we will see in the next chapter how to adapt it to define dual categories for **BBNS** and **BOUS**.

2.5.1 The Dual Adjunction

In this section, we define a functor F : **PreBNS** $\to$ **BOUS**$^{\mathrm{op}}$, and another G : **OUS**$^{\mathrm{op}} \to$ **BBNS** such that, when composed with the inclusions **BOUS** $\hookrightarrow$ **OUS** and **BBNS** $\hookrightarrow$ **PreBNS**, F is a left adjoint to G. We use the same definition to get functors F : **PreBNS**$_{\leq 1} \to$ **BOUS**$^{\mathrm{op}}_{\leq 1}$ and G : **OUS**$^{\mathrm{op}}_{\leq 1} \to$ **BBNS**$_{\leq 1}$, F left adjoint to G. The simplest way to prove this adjunction is to use the unit-counit definition of an adjunction (Theorem 0.4.1 (v)).

In the case of trace-preserving maps, we have already seen F. It is -* : **PreBNS** $\to$ **BOUS**$^{\mathrm{op}}$ from the previous section. We give below the definition for trace-reducing maps, which looks identical to the trace-preserving definition:

$$\begin{aligned} F(X, X_+, \tau) &= (X^*, X_+^*, \tau) \\ F(f) &: F(Y, Y_+, \sigma) \to F(X, X_+, \tau) \\ F(f)(a) &= a \circ f, \end{aligned}$$

where $f : (X, X_+, \tau) \to (Y, Y_+, \sigma)$ and $a \in Y^*$.

To define G, we need a standard theorem.

Theorem 2.5.1 (Ellis [35]). *If (A, A_+, u) is an (archimedean) order-unit space, $(A^*, A_+^*, \mathrm{ev}(u))$ is a (radially compact) Banach base-norm space. The base-norm on A^* agrees with the usual dual norm.*

Proof. In the case that $A \neq 0$, we refer the reader to [6, Theorem 1.19] [4, Theorem II.1.15], each of which proves that the dual is a base-norm space in the Alfsen-Shultz sense, and therefore a base-norm space by Proposition 2.2.20. If we use [6, Theorem 1.19] as a reference, we must additionally use the "norm duality" part of "order and norm duality" in [6, paragraph after Lemma 1.22] and [6, Corollary 1.27]. The fact that the dual is a Banach space follows from the fact that the dual of a normed space is complete, using the fact that the norm coincides with the usual dual norm. In the case that $A = (\{0\}, \{0\}, 0)$ its dual space is $(\{0\}, \{0\}, 0)$, a Banach base-norm space, with empty base. □

We can now define G on objects and maps quite similarly to F:

$$\begin{aligned} G(A, A_+, u) &= (A^*, A_+^*, \mathrm{ev}(u)) \\ G(f) &: G(B, B_+, v) \to G(A, A_+, u) \\ G(f)(\phi) &= \phi \circ f, \end{aligned}$$

where $f : (A, A_+, u) \to (B, B_+, v)$ is a map in **OUS**, corresponding to a map in the opposite direction in **OUS**$^{\mathrm{op}}$, and $\phi \in B^*$.

Proposition 2.5.2. *F and G are functors.*

Proof. We have already shown in the trace-preserving case that F is a functor in Theorem 2.4.18. We also have the general result that $F(X, X_+, \tau)$ is a Banach order-unit space (Proposition 2.4.17), and by Theorem 2.5.1, $G(A, A_+, u)$ is a

Banach base-norm space. We now check that F and G have the correct type on morphisms, for F only in the trace-reducing case.

Let $f : (X, X_+, \tau) \to (Y, Y_+, \sigma)$ be a trace-reducing map, and $a \in Y^*$. We need to show that $F(f)(a) = a \circ f \in X^*$ and that $F(f)$ is positive and unital. Since f is bounded by Proposition 2.2.12, $a \circ f$ is a bounded linear functional $X \to \mathbb{R}$, hence an element of X^*. If we let $g : (A, A_+, u) \to (B, B_+, v)$ be a unital or subunital map, g is bounded by Proposition 1.2.8, it is then the case that $\phi \circ g$ is bounded if ϕ is, and hence is an element of $G(A, A_+, v)$.

The proofs that $F(f)$ and $G(g)$ are positive are nearly identical to each other, so we will only give the proof for $F(f)$ explicitly. We must show that if $a \in Y_+^*$, $F(f)(a) \in X_+^*$. By the definition of the dual cone, this is equivalent to showing that $\forall y \in Y_+ . a(y) \geq 0$ implies $\forall x \in X_+ . F(f)(a)(x) \geq 0$. We can show this as follows. Let $x \in X_+$. Then $F(f)(a)(x) = a(f(x))$. We have that $f(x) \in Y_+$ by positivity of f, and therefore $a(f(x)) \geq 0$ by positivity of a.

To show that $F(f)$ is subunital when f is trace-reducing, we want to show that $F(f)(\sigma) \leq \tau$ in X^*, *i.e.* $\tau - F(f)(\sigma) \in X_+^*$. So let $x \in X_+$, and:

$$(\tau - F(f)(\sigma))(x) = \tau(x) - \sigma(f(x)) \geq 0,$$

by the definition of trace-reducing for f.

To show that $G(g)$ is trace-preserving when g is unital, we must show that $\mathrm{ev}(u) \circ G(g) = \mathrm{ev}(v)$. We do so as follows. Let $\phi \in B^*$. Then

$$\begin{aligned}
(\mathrm{ev}(u) \circ G(g))(\phi) &= \mathrm{ev}(u)(G(g)(\phi)) \\
&= G(g)(\phi)(u) \\
&= \phi(g(u)) && \text{so since } g \text{ is unital} \\
&= \phi(v) = \mathrm{ev}(v)(\phi).
\end{aligned}$$

To show that $G(g)$ is trace-reducing when g is subunital, we want to show that $\mathrm{ev}(u) \circ G(g) \leq \mathrm{ev}(v)$, *i.e.* $\mathrm{ev}(v) - \mathrm{ev}(u) \circ G(g) \in B_+^{**}$. Let $\phi \in B_+^*$, and we have that

$$\begin{aligned}
(\mathrm{ev}(v) - \mathrm{ev}(u) \circ G(g))(\phi) &= \mathrm{ev}(v)(\phi) - \mathrm{ev}(u)(G(g)(\phi)) \\
&= \phi(v) - \mathrm{ev}(u)(\phi \circ g) \\
&= \phi(v) - \phi(g(u)) \\
&= \phi(v - g(u)).
\end{aligned}$$

Since g is subunital, $v - g(u) \in B_+$, and since $\phi \in B_+^*$, this implies $\phi(v - g(u)) \geq 0$. This implies $G(g)$ is trace-reducing.

This establishes that F and G are defined correctly. The fact that they preserve identity arrows follows from the identity law for composition for linear maps, and the fact that they preserve composition follows from the associativity of composition for linear maps. □

Now we can move on to the definition of the unit and counit.

Taking (X, X_+, σ) to be a pre-base-norm space, we define

$$\eta_X : X \to GF(X)$$
$$\eta_X(x)(a) = a(x),$$

where $a \in F(X)$.

For (A, A_+, u) an order-unit space, we want to define $\epsilon_A : FG(A) \to A$ in $\mathbf{OUS}^{\mathrm{op}}$, which means that in **OUS** we define

$$\epsilon_A : A \to FG(A)$$
$$\epsilon_A(a)(\phi) = \phi(a),$$

where $\phi \in G(A)$.

Proposition 2.5.3. *The maps η_X and ϵ_A are well-defined and are natural transformations.*

Proof. We must first show that η_X and ϵ_A are well-defined and are positive. Since the proofs are very similar, we will only state it explicitly for η_X.

We must show that $\eta_X(x) \in GF(X) = X^{**}$, which is to say that if $\phi \in X^*$ and $\|\phi\| \leq 1$, we have $\|\eta_X(x)(\phi)\| \leq \alpha$ for some $\alpha \in [0, \infty)$ (the norms on X^* and X^{**} agree with the usual definition of the dual norms by Proposition 2.2.12, or for ϵ_A, use Theorem 2.5.1 instead). If $\|x\| = 0$, then $x = 0$, and $\eta(x)(\phi) = \phi(0) = 0$ so we can take $\alpha = 0$. So we now assume that $\|x\| > 0$. Then

$$\eta(x)(\phi) = \phi(x) = \phi\left(\|x\|\frac{x}{\|x\|}\right) = \|x\|\phi\left(\frac{x}{\|x\|}\right).$$

Because $\|\phi\| \leq 1$, we have that $\phi\left(\frac{x}{\|x\|}\right) \leq 1$, giving us

$$\eta(x)(\phi) \leq \|x\|.$$

We now show that η_X is positive, *i.e.* that if $x \in X_+$, $\eta_X(x) \in X_+^{**}$. If $a \in X_+^*$, then $\eta_X(x)(a) = a(x)$, which is positive because $x \in X_+$ and $a \in X_+^*$. Since this works for an arbitrary element of X_+^*, we have that $\eta_X(x) \in X_+^{**}$.

We now show separately that η_X is trace-preserving (and hence also trace-reducing) and that ϵ_A is unital (and hence also subunital). We start with η_X. Since $GF(X)$'s trace is $\mathrm{ev}(\tau)$, what we want to show is that $\mathrm{ev}(\tau) \circ \eta_X = \tau$. Taking $x \in X$, we have

$$\mathrm{ev}(\tau)(\eta_X(x)) = \eta_X(x)(\tau) = \tau(x),$$

as required.

For ϵ_A, we have that the unit of $FG(A)$ is $\mathrm{ev}(u)$, so we want to show $\epsilon_A(u) = \mathrm{ev}(u)$. If we take $\phi \in A^*$, we have

$$\epsilon_A(u)(\phi) = \phi(u) = \mathrm{ev}(u)(\phi).$$

Finally, we must show that η_X and ϵ_A define natural transformations. The proof is again very similar in each case so we shall only give the proof for η. Let $f : (X, X_+, \tau) \to (Y, Y_+, \sigma)$ be trace-preserving (or trace-reducing). We want to show that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ GF(X) & \xrightarrow[GF(f)]{} & GF(Y) \end{array}$$

commutes. That is to say, if $x \in X$, and $b \in F(Y) = Y^*$, we want to show that

$$\eta_Y(f(x))(b) = GF(f)(\eta_X(x))(b).$$

We proceed as follows:

$$\begin{aligned} G(F(f))(\eta_X(x))(b) &= (\eta_X(x) \circ F(f))(b) = \eta_X(x)(F(f)(b)) = \eta_X(x)(b \circ f) \\ &= b(f(x)) = \eta_Y(f(x))(b). \end{aligned}$$

We did not use the trace-preservation, so this naturality argument works equally well for trace-reducing maps. Similarly, it also holds for subunital maps. $\square$

We can now finally prove $F \dashv G$.

Theorem 2.5.4. F : **PreBNS** $\to$ $\mathbf{OUS}^{\mathrm{op}}$ *is a left adjoint to* G : $\mathbf{OUS}^{\mathrm{op}} \to$ **PreBNS**, *in both the case of trace-preserving/unital and trace-reducing/subunital maps.*

Proof. We want to show that the following diagrams commute, where $A \in$ **OUS** and $X \in$ **PreBNS**:

$$\begin{array}{ccc} GA & \xrightarrow{\eta_{GA}} & GFGA \\ & {\scriptstyle \mathrm{id}_{GA}}\searrow & \downarrow{\scriptstyle G\epsilon_A} \\ & & GA \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FGFX \\ & {\scriptstyle \mathrm{id}_{FX}}\searrow & \downarrow{\scriptstyle \epsilon_{FX}} \\ & & FX \end{array}$$

The right-hand diagram is written in $\mathbf{OUS}^{\mathrm{op}}$. If it is written in **OUS**, with the arrows turned back to normal, the proof that it commutes is virtually the same as the proof that the left-hand diagram commutes. Therefore we will only give the proof explicitly that the left-hand diagram commutes. The proof for $\mathbf{OUS}_{\leq 1}$ and $\mathbf{PreBNS}_{\leq 1}$ is identical to the proof for **OUS** and **PreBNS** as the definitions of the functors and natural transformations are identical.

With that out of the way, we have to show that for $\phi \in G(A)$, we have

$$G(\epsilon_A)(\eta_{GA}(\phi)) = \phi.$$

We can do this by evaluating the left-hand side at an arbitrary element $a \in A$:

$$\begin{aligned} G(\epsilon_A)(\eta_{GA}(\phi))(a) &= (\eta_{GA}(\phi) \circ \epsilon_A)(a) = \eta_{GA}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) \\ &= \phi(a). \end{aligned}$$

$\square$

As with any adjunction, we can consider those objects such that the unit and counit are isomorphisms. We can define *reflexive order-unit spaces* to be order-unit spaces A such that ϵ_A is an isomorphism of order-unit spaces and and *reflexive base-norm spaces* to be pre-base-norm spaces X such that η_X is an isomorphism of pre-base-norm spaces. We can define **RBNS** to be the full subcategory of **PreBNS** on reflexive base-norm spaces and **ROUS** to be the full subcategory of **OUS** on reflexive order-unit spaces. Recall that a normed space E is called *reflexive* if the evaluation map $E \to E^{**}$ is a bijection[24, III Definition 11.2].

Proposition 2.5.5. *A pre-base-norm space is reflexive iff its underlying normed space is a reflexive and it is a base-norm space. An order-unit space is reflexive iff its underlying normed space is reflexive. The functors F and G, when restricted to* **RBNS** *and* **ROUS** *respectively, form an adjoint equivalence* $\mathbf{RBNS} \simeq \mathbf{ROUS}^{\mathrm{op}}$.

Proof. On the underlying normed spaces, the maps ϵ_A and η_E are the evaluation maps. Therefore reflexivity of the underlying normed spaces is necessary in both cases. As dual cones are always weakly closed (Lemmas 0.3.5 and 0.3.7) they are closed in the finer norm topology as well, it is also necessary that a reflexive pre-base-norm space be a base-norm space, otherwise the inverse of η_E would not be positive. We have therefore shown the necessity in both cases.

For sufficiency, observe first that any bijective unital map of order-unit spaces has unital inverse, and every bijective trace-preserving map of pre-base-norm spaces has trace-preserving inverse. Both a base-norm space (by definition) and an order-unit space (by Lemma A.5.3) have a closed positive cone. We therefore only need to show that for a reflexive Banach space E with a closed cone E_+, the inverse of the evaluation mapping $E^{**} \to E$ is positive, where E^{**} is given the double dual cone. This is equivalent to showing that $\mathrm{ev}^{-1}(E_+^{**}) \subseteq E_+$, as we already know the opposite inclusion holds by Proposition 2.5.3. So let $x \in E$ be such that $\mathrm{ev}(x) \in E_+^{**}$. Then by expanding the definition of the dual cone, we have that for all $\phi \in E_+^*$, $\mathrm{ev}(x)(\phi) \geq 0$. As $\mathrm{ev}(x)(\phi) = \phi(x)$, this is equivalent to $\phi(x) \geq 0$. Then Lemma 0.3.15 gives us $x \in E_+$ because it is closed.

We then have that F and G form an adjoint equivalence $\mathbf{ROUS}^{\mathrm{op}} \simeq \mathbf{RBNS}$ by the "unity of opposites" – the triangle laws for an adjunction imply that if η_E is an isomorphism, $\epsilon_{F(E)}$ is too, and similarly ϵ_A an isomorphism implies $\eta_{G(A)}$ is an isomorphism [81, Part 0, Proposition 4.2]. ☐

A normed space is reflexive iff its unit ball is compact in the weak topology[24, V Theorem 4.2]. Therefore every finite-dimensional normed space is reflexive because the unit ball is compact by the Heine-Borel theorem. So the finite dimensional base-norm spaces (excluding pre-base-norm spaces) and finite dimensional order-unit spaces are all reflexive. For C*-algebras, we have a converse – every reflexive C*-algebra is finite dimensional [128, I.11 Exercise 2]. However, there are reflexive infinite-dimensional order-unit spaces such as the *spin factors*[7, Proposition 3.38]. The corresponding base-norm spaces are those arising from taking the unit ball of a Hilbert space as an element of **CBConv** [7, Proposition 5.51]. However, this duality turns out not to be general enough to include all the useful examples and so we come to a pair of dualities in chapter 3 generalizing this adjunction.

Chapter 3

Smith Spaces

3.1 Introduction

The purpose of this chapter is to show how the dual adjunction between base-norm spaces and order-unit spaces in the previous chapter can be converted into two dualities. In the previous chapter we saw that we could produce a duality through the use of reflexive spaces, so in this chapter we first look at a way to make every Banach space "reflexive", by considering a different dual topology. This leads us to consider Smith spaces. A space E is a *Smith space* if it is linearly homeomorphic to the dual space of some Banach space F, given the *bounded weak-* topology* or equivalently the topology of uniform convergence on precompact sets. The term was introduced by Akbarov [2, Example 4.6], naming them after M. F. Smith who published a paper on Pontryagin duality for Banach spaces[123, Theorem 2]. Akbarov gives in [2, Proposition 4.7] and [3, Proposition 1.2] an intrinsic characterization of Smith spaces as locally convex spaces E satisfying three criteria:

(i) E is complete.

(ii) E is compactly generated.

(iii) There is a compact absolutely convex set in E that absorbs every compact subset (a *universal compact set*).

In this chapter we show that the first condition can be dropped and the other two relaxed so a space can be confirmed to be Smith more easily (Proposition 3.2.9). We then prove a categorical equivalence (Theorem 3.2.22)

$$\mathbf{Ban}^{\mathrm{op}} \simeq \mathbf{Smith},$$

though this is known, with Akbarov's definitions, from [2][1]. In fact, we would be able to do this using the more easily defined weak-* topology instead of the

[1]Combine the fact that the dual space functor on stereotype spaces is a contravariant equivalence with the fact that the dual of a Banach space is Smith and vice-versa.

bounded weak-* topology, but we know of no characterization of such spaces, except by first characterizing the bounded weak-* topology. Another dual category to Banach spaces is the category of Waelbroeck spaces, as used in [21, Chapter I, Theorem 2.8]. These spaces only put a topology on the unit ball and so avoid having to make a distinction between the weak-* and bounded weak-* topologies, although the difficulties are shifted elsewhere as one does not, *prima facie*, have a linear topology.

We then define Smith base-norm and order-unit spaces, and prove

$$\mathbf{SBNS} \simeq \mathbf{BOUS}^{\mathrm{op}}$$
$$\mathbf{BBNS} \simeq \mathbf{SOUS}^{\mathrm{op}}.$$

We can extend these equivalences to two squares of equivalences:

$$\begin{array}{ccc} \mathbf{BOUS}^{\mathrm{op}} & \underset{F^{\beta}}{\overset{G^{\sigma}}{\rightleftarrows}} & \mathbf{SBNS} \\ \mathcal{T}\uparrow\downarrow [0,1]_{-} & & \mathrm{Emb}\uparrow\downarrow B \\ \mathbf{BEMod}^{\mathrm{op}} & \underset{\mathrm{CAff}(-,[0,1])}{\overset{\mathrm{Stat}}{\rightleftarrows}} & \mathbf{CCL} \end{array} \tag{3.1}$$

$$\begin{array}{ccc} \mathbf{SOUS}^{\mathrm{op}} & \underset{F^{\sigma}}{\overset{G^{\beta}}{\rightleftarrows}} & \mathbf{BBNS} \\ \mathcal{T}\uparrow\downarrow [0,1]_{-} & & \mathrm{Emb}\uparrow\downarrow B \\ \mathbf{CEMod}^{\mathrm{op}} & \underset{\mathrm{BAff}(-,[0,1])}{\overset{\mathrm{CStat}}{\rightleftarrows}} & \mathbf{CBConv} \end{array} \tag{3.2}$$

Where **CEMod** and **BEMod** are categories of effect modules to be used as predicates, and **CBConv** and **CCL** are categories of convex sets to be used as state spaces. The square (3.1) can be viewed as a summary of Kadison duality (see [59] and [70]), the equivalence between **BOUS** and **BEMod** (see [63, Proposition 11] and [48]), and the fact that a base-norm space is a dual space iff it can be given a locally convex topology in which its base is compact [35, Theorem 3]. The square (3.2), on the other hand, is mostly original, the only preceding result being [35, Theorem 6], that an order-unit space is the dual of a base-norm space iff its unit interval is compact in some locally convex topology. We can use these squares and the relationship between C*-algebras and order-unit spaces to produce a state and effect triangle for each of $\mathbf{C^*Alg}_{\mathrm{PU}}$ and $\mathbf{W^*Alg}_{\mathrm{PU}}$.

In the next chapter we will see how to express the category **CCL** in terms of Eilenberg-Moore algebras, and so inherit the convenient properties thereof. In this chapter, we do show that the category **CBConv** is a reflective subcategory of $\mathcal{EM}(\mathcal{D}_\infty)$ and $\mathcal{EM}(\mathcal{D})$. This shows how we can take the bare minimum for a structure of probabilistic mixtures on a set and "freely" construct a Banach base-norm space from it.

In Section 3.5 we show what happens if we combine the dualities in this chapter with the adjunction in Subsection 2.5.1. The resulting adjunctions are related to enveloping W*-algebras and also to Semadeni's universal compactification for convex sets.

3.2 Smith Spaces

Let E be a normed space and E^* its dual space. We have already seen that E^* can be given the dual norm, which defines one topology, and it can also be given the weak-* (or $\sigma(E^*, E)$) topology. We now concern ourselves with a third topology, in between these, the *bounded weak-** topology (see [32, Definition V.5.3, Corollary V.5.5]). This topology is defined to be the finest topology agreeing with $\sigma(E^*, E)$ on bounded sets, *i.e.* a set $O \subseteq E$ is open if for all $\alpha \in (0, \infty)$ there exists a $\sigma(E^*, E)$-open O' such that

$$O \cap \alpha U = O' \cap \alpha U.$$

On the dual of a Banach space it is also the same topology as the polar topology for compact subsets of E ([8, Chapter 1, Theorem 2.2]). Smith spaces are related to the circle of ideas around the Krein-Šmulian theorem, the Banach-Dieudonné theorem and Grothendieck's completeness theorem.

A *barrel* in a locally convex space $(E, \mathcal{T})$ is a subset $B \subseteq E$ that is absolutely convex, closed, and absorbent. To avoid confusion, we state here that a *barrelled* space is one in which every barrel is a zero neighbourhood, but that every locally convex space contains several barrels, whether or not it is barrelled.

We now give our definition of a Smith space. A *Smith space* $(E, \mathcal{T}, B)$ is a locally convex space E, the topology being $\mathcal{T}$, and a compact barrel B, such that $\mathcal{T}$ is the finest topology agreeing with $\mathcal{T}$ on all the subsets αB for $\alpha \in \mathbb{R}_{>0}$. The category **Smith** of Smith spaces has continuous linear maps as morphisms, and the category $\mathbf{Smith}_1$ is the subcategory of maps between Smith spaces $f : (E, \mathcal{T}, B) \to (F, \mathcal{S}, C)$ such that $f(B) \subseteq C$.

At this stage, we can show that closed subspaces of Smith spaces are Smith.

Lemma 3.2.1. *If $(E, \mathcal{T}, B)$ is a Smith space, $F \subseteq E$ a closed linear subspace, then $(F, \mathcal{T}|_F, B \cap F)$ is a Smith space.*

Proof. We first show that $B \cap F$ is a compact barrel. We use the name $C = B \cap F$. We have that $\mathcal{T}|_F$ agrees with $\mathcal{T}$ on F, so C is $\mathcal{T}|_F$-compact because C is a closed subspace of a compact space B. As C is the intersection of two absolutely convex sets, it is absolutely convex. To show that it is absorbent, let $x \in F$. As $x \in E$, there exists $\alpha \in \mathbb{R}_{>0}$ such that $x \in \alpha B$. Therefore

$$x \in (\alpha B) \cap F = \alpha B \cap \alpha F = \alpha(B \cap F) = \alpha C.$$

To show $(F, \mathcal{T}|_F, C)$ is Smith, we only need to show that any set $U \subseteq F$ such that for all $\alpha \in \mathbb{R}_{>0}$ there exists $U_\alpha \in \mathcal{T}|_F$ $U \cap \alpha C = U_\alpha \cap \alpha C$, then $U \in \mathcal{T}|_F$. So let U be such a set. As $U_\alpha \in \mathcal{T}|_F$, there exists $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap F$. We first show that

$$(V_\alpha \cup (E \setminus F)) \cap \alpha B = (U \cup (E \setminus F)) \cap \alpha B \tag{3.3}$$

for all $\alpha \in \mathbb{R}_{>0}$.

We have

$$\begin{aligned}(V_\alpha \cup (E \setminus F)) \cap \alpha B &= ((V_\alpha \cap F) \cup (E \setminus F)) \cap \alpha B \\ &= (V_\alpha \cap F \cap \alpha B) \cup ((E \setminus F) \cap \alpha B) \\ &= (U_\alpha \cap \alpha C) \cup ((E \setminus F) \cap \alpha B) \\ &= (U \cap \alpha C) \cup ((E \setminus F) \cap \alpha B) \\ &= (U \cap F \cap \alpha B) \cup ((E \setminus F) \cap \alpha B) \\ &= (U \cup (E \setminus F)) \cap \alpha B,\end{aligned}$$

proving (3.3).

As F is closed, $E \setminus F$ is $\mathcal{T}$-open, so $V_\alpha \cup (E \setminus F)$ is a $\mathcal{T}$-open set. As E is a Smith space, (3.3) shows that $U \cup (E \setminus F)$ is $\mathcal{T}$-open. Therefore

$$U = (U \cup (E \setminus F)) \cap F$$

is a $\mathcal{T}|_F$-open set, so $(F, \mathcal{T}|_F, C)$ is a Smith space. □

From what we have proven so far, it is not yet clear that there are any useful Smith spaces, or that Smithness can be verified usefully in practice. The rest of this section is dedicated to showing that these statements are true.

Lemma 3.2.2. *Let $(E, \mathcal{T}, B)$ be a locally convex space $(E, \mathcal{T})$ and a compact barrel $B \subseteq E$. The set $B^\circ \subseteq E^*$ is radially compact and absorbent, and therefore defines a norm $\|\text{-}\|_{B^\circ}$ on E^*, of which B° is the closed unit ball.*

Proof. We first show that B° is radially bounded. Suppose for a contradiction that B° is radially unbounded. Then it contains an element $\phi \neq 0$ such that $n\phi \in B^\circ$ for all $n \in \mathbb{N}$. Since $B^\circ = B^{|\circ|}$ (Lemma 0.3.6), we see that for all $n \in \mathbb{N}$ and $x \in B$

$$|\langle n\phi, x\rangle| \leq 1 \Leftrightarrow |n\phi(x)| \leq 1 \Leftrightarrow |\phi(x)| \leq \frac{1}{n}$$

Therefore $\phi(x) = 0$ for all $x \in B$. Since B is absorbent, its span is all of E, so $\phi = 0$, a contradiction.

Since B° is a polar, it is $\sigma(E^*, E)$-closed, therefore the intersection of any line with B° is closed, so it is radially compact.

To show that B° is absorbent, let $\phi \in E^*$. As ϕ is continuous, there is an absolutely convex 0-neighbourhood $U \subseteq E$ such that $\phi(E) \subseteq (-1, 1)$. Since B is compact, it is bounded (Lemma 0.1.14), so there is an $\alpha \in \mathbb{R}_{>0}$ such that $B \subseteq \alpha U$. Therefore $\alpha^{-1}B \subseteq U$ and so $\phi(\alpha^{-1}B) \subseteq (-1, 1)$. This implies $\phi \in (\alpha^{-1}B)^\circ = \alpha B^\circ$ (Lemma 0.3.11 (ii)).

By Lemma 0.1.5 $\|\text{-}\|_{B^\circ}$ is a norm, and by Lemma 0.1.7, B° is the closed unit ball of $\|\text{-}\|_{B^\circ}$. □

In fact, as B is itself radially compact, $\|\text{-}\|_B$ is a norm.

Lemma 3.2.3. *Let $(E, \mathcal{T}, B)$ be a locally convex space with compact barrel B. The topology defined by the norm $\|\text{-}\|_B$ is finer than $\mathcal{T}$.*

Proof. As the $\|\text{-}\|_B$ topology and $\mathcal{T}$ are both locally convex topologies, it suffices to show that every 0-neighbourhood for $\mathcal{T}$ is a 0-neighbourhood for $\|\text{-}\|_B$. So let N be a 0-neighbourhood for $\mathcal{T}$, and $U \subseteq N$ an open subset of N containing 0. As all compact sets are bounded (Lemma 0.1.14), we have that there exists $\alpha \in \mathbb{R}_{>0}$ such that $B \subseteq \alpha U$. Therefore $\alpha^{-1}B \subseteq U \subseteq N$, so N is a 0-neighbourhood for $\|\text{-}\|_B$. □

Lemma 3.2.4. *If $(E, \mathcal{T}, B)$ is a locally convex space with B a compact barrel, the usual pairing $\langle \text{-}, \text{-} \rangle$ between E and E^* is separately continuous for $\mathcal{T}$ and $\|\text{-}\|_{B^\circ}$.*

Proof.

- For all $\phi \in E^*$, $\langle \text{-}, \phi \rangle : (E, \mathcal{T}) \to \mathbb{R}$ is continuous:

  Since $\langle \text{-}, \phi \rangle = \phi$, this follows from the definition of E^* as the continuous dual.

- For all $x \in E$, $\langle x, \text{-} \rangle : (E^*, \|\text{-}\|_{B^\circ}) \to \mathbb{R}$ is continuous:

  As E^* is normed, we only need to show that $\langle x, \text{-} \rangle$ is bounded for each $x \in E$, *i.e.* that there exists some $\alpha \in \mathbb{R}_{>0}$ such that $\langle x, B^\circ \rangle \subseteq [-\alpha, \alpha]$. Since B is absorbent, there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha^{-1}x \in B$. We know by Lemma 0.3.6 that $|\phi(\alpha^{-1}x)| \leq 1$ for all $\phi \in B^\circ$. Therefore $|\langle x, \phi \rangle| = |\phi(x)| \leq \alpha$ for all $\phi \in B^\circ$, which gives us that $\langle x, B^\circ \rangle \subseteq [-\alpha, \alpha]$, as required. □

We need the following small lemma about the Minkowski functional only for the next proposition.

Lemma 3.2.5. *Let B be an absolutely convex, radially compact and absorbent subset of a real vector space E. Then for all $\alpha \in \mathbb{R}_{>0}$ and $y \in E$*

$$\alpha B \subseteq (\|y\|_B + \alpha)B + y.$$

Proof. If $x \in \alpha B$, then $\|x\|_B \leq \alpha$. This implies $\|x - y\|_B \leq \|x\|_B + \|y\|_B = \|y\|_B + \alpha$. The radial compactness implies $x - y \in (\|y\|_B + \alpha)B$ (Lemma 0.1.7), and therefore $x \in (\|y\|_B + \alpha)B + y$. □

In the following proposition we show how to redefine the topology on any locally convex space with a compact barrel.

Proposition 3.2.6. *Let $(E, \mathcal{T}, B)$ be a locally convex space with B a compact barrel. Then:*

(i) $\mathcal{T}_b = \{U \subseteq E \mid \forall \alpha \in \mathbb{R}_{>0}. \exists U_\alpha \in \mathcal{T}. U \cap \alpha B = U_\alpha \cap \alpha B\}$ *is a topology.*

(ii) $\mathcal{T}_b$ *is the finest topology agreeing with* $\mathcal{T}$ *on* αB *for all* $\alpha \in \mathbb{R}_{>0}$.

(iii) $\mathcal{T}_b$ *is Hausdorff, and if* $\mathcal{N}_x$ *is the neighbourhood filter at* x*, then* $\mathcal{N}_x = \mathcal{N}_0 + x$*.*

Proof.

(i) We see that $\emptyset \in \mathcal{T}_b$ and $E \in \mathcal{T}_b$ because they are both in $\mathcal{T}$ and so $\emptyset \cap \alpha B = \emptyset \cap \alpha B$ and likewise for E.

Let $U, V \in \mathcal{T}_b$, with U_α and V_α defined as expected. We see that

$$\begin{aligned}(U \cap V) \cap \alpha B = (U \cap \alpha B) \cap (V \cap \alpha B) &= (U_\alpha \cap \alpha B) \cap (V_\alpha \cap \alpha B)\\ &= (U_\alpha \cap V_\alpha) \cap \alpha B,\end{aligned}$$

so $U \cap V \in \mathcal{T}_b$.

If we let $(U_i)_{i\in I}$ be a family of sets in $\mathcal{T}_b$, with $U_{i,\alpha}$ the corresponding families of elements of $\mathcal{T}$, then

$$\left(\bigcup_{i\in I} U_i\right) \cap \alpha B = \bigcup_{i\in I}(U_i \cap \alpha B) = \bigcup_{i\in I}(U_{i,\alpha} \cap \alpha B) = \left(\bigcup_{i\in I} U_{i,\alpha}\right) \cap \alpha B,$$

so $\mathcal{T}_b$ is closed under unions too, and is therefore a topology.

(ii) There are two parts, the first is showing $\mathcal{T}_b$ is finer than any topology agreeing with $\mathcal{T}$ on each set αB where $\alpha \in \mathbb{R}_{>0}$. The second is to show that $\mathcal{T}_b$ agrees with $\mathcal{T}$ on each αB.

Let $\mathcal{S}$ be a topology on E that agrees with $\mathcal{T}$ on each set αB. We need to show that $\mathcal{S} \subseteq \mathcal{T}_b$. Let $U \in \mathcal{S}$. Since $\mathcal{S}$ agrees with $\mathcal{T}$ on αB, there is some $U_\alpha \in \mathcal{T}$ such that $U_\alpha \cap \alpha B = U \cap \alpha B$. We have therefore shown $U \in \mathcal{T}_b$. It follows that $\mathcal{T}_b$ is also finer than $\mathcal{T}$.

We must now show that $\mathcal{T}_b$ agrees with $\mathcal{T}$ on each αB. Since $\mathcal{T}_b$ is finer than $\mathcal{T}$, $\mathcal{T}_b|_{\alpha B}$ is finer than $\mathcal{T}|_{\alpha B}$. To prove the above, we must show that $\mathcal{T}|_{\alpha B}$ is finer than $\mathcal{T}_b|_{\alpha B}$. If $U' \in \mathcal{T}_b|_{\alpha B}$, then $U' = U \cap \alpha B$ for some $U \in \mathcal{T}_b$. Then $U \cap \alpha B = U_\alpha \cap \alpha B$ for some $U_\alpha \in \mathcal{T}$, so $U' \in \mathcal{T}|_{\alpha B}$.

(iii) Since we showed in (ii) that $\mathcal{T}_b$ is finer than $\mathcal{T}$, it is Hausdorff because $\mathcal{T}$ is. To prove the rest of the statement, we will first show that $\mathcal{N}_x + y \subseteq \mathcal{N}_{x+y}$ in $\mathcal{T}_b$. If $N \in \mathcal{N}_x$, we have that there is a $U \in \mathcal{T}_b$ such that $x \in U \subseteq N$. We see that $x + y \in U + y \subseteq N + y$, so we have proven the inclusion of neighbourhoods if we can show that $U + y \in \mathcal{T}_b$. We define $(U + y)_\alpha = U_{\alpha+\|y\|_B} + y$. Then we have

$$\begin{aligned}&U \cap (\alpha + \|y\|_B)B = U_{\alpha+\|y\|_B} \cap (\alpha + \|y\|_B)B\\ &\Rightarrow (U + y) \cap ((\alpha + \|y\|_B)B + y) = (U + y)_\alpha \cap (\alpha B + y)\\ &\Rightarrow (U + y) \cap \alpha B = (U + y)_\alpha \cap \alpha B,\end{aligned}$$

by Lemma 3.2.5. This shows $U + y \in \mathcal{T}_b$.

Now that we have established that $\mathcal{N}_x + y \subseteq \mathcal{N}_{x+y}$, we have $\mathcal{N}_0 + x \subseteq \mathcal{N}_x$, and $\mathcal{N}_x + -x \subseteq \mathcal{N}_0$, and by adding x to the second one we get $\mathcal{N}_x \subseteq \mathcal{N}_0 + x$. □

We now show how $\mathcal{T}$ and $\mathcal{T}_b$ relate to the weak topology, $\sigma(E, E^*)$.

Lemma 3.2.7. *Let $(E, \mathcal{T}, B)$ be a locally convex space with compact barrel B. Then $\mathcal{T}$ and $\sigma(E, E^*)$ agree on each set αB for $\alpha \in \mathbb{R}_{>0}$, and so $\mathcal{T}_b = \sigma(E, E^*)_b$.*

Proof. Since $\alpha \cdot$- is continuous, αB is compact for all $\alpha \in \mathbb{R}_{>0}$. We also have that by definition $\sigma(E, E^*)$ is coarser than $\mathcal{T}$, so the map $\mathrm{id} : (E, \mathcal{T}) \to (E, \sigma(E, E^*))$ is continuous. Therefore αB is compact in $\sigma(E, E^*)$, and it is also Hausdorff because $\sigma(E, E^*)$ is. Therefore $\mathrm{id} : (\alpha B, \mathcal{T}|_B) \to (\alpha B, \sigma(E, E^*)|_B)$ is a continuous bijection of compact Hausdorff spaces, and therefore a homemorphism. Then $\mathcal{T}_b = \sigma(E, E^*)_b$ follows from Proposition 3.2.6 (ii). □

We can conclude from the above that the topologies admitting a compact barrel are quite restricted in how they can behave.

The following definition and proposition draw on [32, V.5.4-5], which was not sufficiently general for our purposes.

If we have $(E, \mathcal{T}, B)$, a locally convex space with compact barrel E, and $(\phi_i)_{i \in \mathbb{N}}$ is a sequence in E^* converging to 0 with respect to $\|\text{-}\|_{B^\circ}$. Define

$$N_{(\phi_i)} = \{x \in E \mid \forall i \in \mathbb{N}. |\phi_i(x)| < 1\}$$

Proposition 3.2.8. *Given $(E, \mathcal{T}, B)$, the sets of the form $N_{(\phi_i)}$ form a base for the neighbourhood filter of 0 in $(E, \mathcal{T}_b)$.*

Proof. We use Lemma 3.2.7 to reduce the problem to showing that the family of sets $N_{(\phi_i)}$ is a neighbourhood base for 0 in $(E, \sigma(E, E^*)_b)$.

- $\{N_{(\phi_i)} \mid \phi_i \to 0 \text{ in } (E^*, \|\text{-}\|_{B^\circ})\}$ is a filter base:

  Recall the definition of a filter base from [19, I.6.3]: A set B of subsets of E is a filter base if the intersection of two sets from B contains a set from B and B is nonempty and does not contain the empty set. To show the first property, we need to prove one thing first. Let $(\phi_i)_{i \in \mathbb{N}}$ and $(\phi'_i)_{i \in \mathbb{N}}$ be sequences converging to zero in E^*. Define

  $$\psi_{2i} = \phi_i \qquad\qquad \psi_{2i+1} = \phi'_i.$$

  Then we show that $\psi_i \to 0$. Let $\epsilon \in \mathbb{R}_{>0}$. We have a $n, n' \in \mathbb{N}$ such that for all $i \geq n$ $\|\phi_i\| < \epsilon$ and for all $i \geq n'$, $\|\phi'_i\| < \epsilon$. Take $m = 2 \max n, n' + 1$. If $i \geq m$ and is odd, then $\|\psi_i\| = \|\phi_{\frac{i-1}{2}}\| < \epsilon$ because $\frac{i-1}{2} \geq n'$. Likewise, if i is even, $\|\psi_i\| = \|\phi_{\frac{i}{2}}\| < \epsilon$ because $\frac{i}{2} \geq n$. Therefore $N_{(\psi_i)}$ fits the definition. Now

  $$\begin{aligned} N_{(\phi_i)} \cap N_{(\phi'_i)} &= \{x \in E \mid \forall i \in \mathbb{N}. |\phi_i(x)| < 1 \text{ and } |\phi'_i(x)| < 1\} \\ &= \{x \in E \mid \forall i \in \mathbb{N}. |\psi_{2i}(x)| < 1 \text{ and } |\psi_{2i+1}(x)| < 1\} \\ &= \{x \in E \mid \forall i \in \mathbb{N}. |\psi_i(x)| < 1\} \\ &= N_{(\psi_i)}. \end{aligned}$$

We have therefore verified the first property. To show that there always exists a set of the form $N_{(\phi_i)}$, we can take $\phi_i = 0$ for all $i \in \mathbb{N}$. Then $N_{(\phi_i)} = E$. To show that all the $N_{(\phi_i)}$ are nonempty, we observe that however (ϕ_i) is defined, we always have $|\phi(0)| = 0 < 1$, so $0 \in N_{(\phi_i)}$ for all (ϕ_i).

- $N_{(\phi_i)}$ is a $\sigma(E, E^*)_b$-open neighbourhood of 0:

  We just showed that $N_{(\phi_i)}$ always contains 0, so we only need to show that it is $\sigma(E, E^*)$-open. Recall from the definition of $\sigma(E, E^*)$ that $(N_\phi)_{\phi \in E^*}$ is a subbase of open 0-neighbourhoods (see (2)):

$$N_\phi = \{x \in E \mid |\phi(x)| < 1\}.$$

  Observe that $N_{(\phi_i)} = \bigcap_{i \in \mathbb{N}} N_{\phi_i}$. We can prove $N_{(\phi_i)}$ is $\sigma(E, E^*)_b$-open by finding, for each $\alpha \in \mathbb{R}_{>0}$, a finite set $\{\phi_1, \dots, \phi_n\} \subseteq E^*$ such that $N_{(\phi_i)} \cap \alpha B = \bigcap_{j=1}^{n} N_{\phi_j} \cap \alpha B$.

  To do this, we start with the fact that for all $\epsilon \in \mathbb{R}_{>0}$, there is an $n_\epsilon \in \mathbb{N}$ such that for all $i \geq n_\epsilon$, $\|\phi_i\|_{B^\circ} < \epsilon$. If $i \geq n_{(2\alpha)^{-1}}$, then $\|\phi_i\|_{B^\circ} < (2\alpha)^{-1}$, so since B° is the closed unit ball (Lemma 3.2.2), $\phi_i \in (2\alpha)^{-1} B^\circ = \frac{1}{2}\alpha B^\circ$ (Lemma 0.3.11 (ii)). So we have that for all $i \geq n_{(2\alpha)^{-1}}$ and all $x \in \alpha B$

$$|2\phi_i(x)| \leq 1 \Leftrightarrow |\phi_i(x)| \leq \frac{1}{2} \Rightarrow |\phi_i(x)| < 1 \Leftrightarrow x \in N_{\phi_i}.$$

  So $\alpha B \subseteq N_{\phi_i}$ for $i \geq n_{(2\alpha)^{-1}}$.

  Let $m = n_{(2\alpha)^{-1}}$, and

$$\begin{aligned} N_{(\phi_i)} \cap \alpha B = \bigcap_{i=1}^{\infty} N_{\phi_i} \cap \alpha B &= \left(\bigcap_{i=1}^{m} N_{\phi_i} \alpha B \right) \cap \bigcap_{i=m+1}^{\infty} \alpha B \\ &= \left(\bigcap_{i=1}^{m} N_{\phi_i} \right) \cap \alpha B. \end{aligned}$$

  Now, $\bigcap_{i=1}^{m} N_{\phi_i}$ is $\sigma(E, E^*)$-open, and since this can be done for all $\alpha \in \mathbb{R}_{>0}$, we have shown $N_{(\phi_i)}$ is $\sigma(E, E^*)_b$-open.

- The sets $N_{(\phi_i)}$ generate the neighbourhood filter for 0, *i.e.* for each $\sigma(E, E^*)_b$ 0-neighbourhood N, there is a (ϕ_i) such that $\phi_i \to 0$ in $(E^*, \|\text{-}\|_{B^\circ})$ such that $N_{(\phi_i)} \subseteq N$:

  Let N be a $\sigma(E, E^*)$ 0-neighbourhood, and $U \subseteq N$ its interior, which is necessarily an open 0-neighbourhood. We constuct $(\phi_i)_{i \in \mathbb{N}}$ inductively. We define two countable families of finite subsets of E^*, which we call $(X_i)_{i \in \mathbb{N}}$ and $(Y_i)_{i \in \mathbb{N}}$, such that ${X_i}^{|\circ|} \cap iB \subseteq U$, $X_{i+1} = X_i \cup Y_{i+1}$, and for $i > 1$, $Y_{i+1} \subseteq \frac{1}{i} B^{|\circ|}$.

We first observe that $U \cap B = U_1 \cap B$ for some U_1 that is $\sigma(E, E^*)$-open. We therefore have that there are $\phi_1, \ldots, \phi_n \in E^*$ such that $\bigcap_{i=1}^n N_{\phi_i} \subseteq U_1$. We define $Y_1 = X_1 = \{2\phi_1, \ldots, 2\phi_n\}$. We then observe that

$$\begin{aligned} {X_1}^{|\circ|} &= \{x \in X \mid \forall i \in \{1, \ldots, n\}. |2\phi_i(x)| \leq 1\} \\ &= \left\{ x \in X \,\middle|\, \forall i \in \{1, \ldots, n\}. |\phi_i(x)| \leq \frac{1}{2} \right\} \\ &\subseteq \{x \in X \mid \forall i \in \{1, \ldots, n\}. |\phi_i(x)| < 1\} \\ &= \bigcap_{i=1}^{n} N_{\phi_i} \subseteq U_1, \end{aligned}$$

so ${X_1}^{|\circ|} \cap B = U_1 \cap B \subseteq U$.

The inductive step proceeds as follows. Assume that there is an $X_i \subseteq E^*$ such that ${X_i}^{|\circ|} \cap iB \subseteq U$. We only use this part of the inductive hypothesis. We show that there exists a finite $Y_{i+1} \subseteq \frac{1}{i} B^{|\circ|}$ such that

$$(X_i \cup Y_{i+1})^{|\circ|} \cap (i+1)B \subseteq U$$

by contradiction. Assume for a contradiction that if $Y \subseteq \frac{1}{i} B^{|\circ|}$ is finite, then $(X_i \cup Y)^{|\circ|} \cap (i+1)B \not\subseteq U$. We define $\mathcal{F}$ to be the set of all subsets of E^* of the form $(X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U)$, with $Y \subseteq \frac{1}{i} B^{|\circ|}$. By the assumption, $\mathcal{F}$ consists of non-empty sets. Now, $(X_i \cup Y)^{|\circ|}$ is $\sigma(E, E^*)$-closed because it is an absolute polar, and $(i+1)B \cap E \setminus U$ is $\sigma(E, E^*)$-closed because U is $\sigma(E, E^*)_b$-open. So $\mathcal{F}$ consists of closed subsets of $(i+1)B$. If Y and Y' are such that $(X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U) \in \mathcal{F}$ and likewise for Y', then we have

$$\begin{aligned} &(X_i \cup Y)^{|\circ|} \cap (i+1)B \cap (E \setminus U) \cap (X_i \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \\ &= (X_i \cup Y)^{|\circ|} \cap (X_i \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \\ &= (X_i \cup Y \cup Y')^{|\circ|} \cap (i+1)B \cap (E \setminus U) \qquad \text{Lemma 0.3.11 (i).} \end{aligned}$$

The set $Y \cup Y_i$ is also a finite subset of $\frac{1}{i} B^{|\circ|}$, so we have shown that $\mathcal{F}$ is closed under finite intersections, and any finite intersection of sets in $\mathcal{F}$ is non-empty. By the intersection formulation of compactness [19, I.9.1 (C")], $\bigcap \mathcal{F}$ is not empty. Let $x \in \bigcap \mathcal{F}$. By Lemma 0.3.11(iii), $(X_i \cup Y)^{|\circ|} \subseteq {X_i}^{|\circ|}$ for any set Y, so $x \subseteq {X_i}^{|\circ|} \cap (i+1)B \cap (E \setminus U)$. We also have that for all $\phi \in \frac{1}{i} B^{|\circ|}$, $x \in \{\phi\}^{|\circ|}$, which is to say, $|\phi(x)| \leq 1$.

Therefore

$$\begin{aligned}
x &\in \left\{ x' \in X \,\middle|\, \forall \phi \in \frac{1}{i} B^{|\circ|} . |\phi(x)| \leq 1 \right\} \\
&= \left(\frac{1}{i} B^{|\circ|} \right)^{|\circ|} \\
&= (iB)^{|\circ||\circ|} && \text{Lemma 0.3.11 (ii)} \\
&= iB && \text{Corollary 0.3.12,}
\end{aligned}$$

so in fact, $x \in {X_i}^{|\circ|} \cap iB \cap E \setminus U$, which contradicts the inductive hypothesis that ${X_i}^{|\circ|} \cap iB \subseteq U$.

We then define Y_i to be a finite subset of $\frac{1}{i} B^{|\circ|}$ that we have just shown to exist, and define $X_{i+1} = X_i + Y_{i+1}$. Then we now have ${X_{i+1}}^{|\circ|} \cap (i+1)B \subseteq U$, as required. This finishes the inductive construction.

Define $(\phi_i)_{i \in \mathbb{N}}$ to enumerate the elements of the Y_i in increasing order of i.

To see that $\phi_i \to 0$ for $\|\text{-}\|_{B^{|\circ|}}$, let $\epsilon \in \mathbb{R}_{>0}$. There is a smallest m such that $\frac{1}{m} < \epsilon$. For all $i \geq m$, any $\phi \in Y_i$ is in $\frac{1}{m} B^{|\circ|}$. We define $n = \sum_{j=1}^{m} |Y_i| + 1$. Then if $i \geq n$ we have $\phi_i \in \frac{1}{m} B^{|\circ|}$, so $\|\phi_i\|_{B^{|\circ|}} \leq \frac{1}{m} < \epsilon$, proving convergence to 0.

All that remains is to show that $N_{(\phi_i)} \subseteq U$, as $U \subseteq N$. From the definitions, we have $N_{(\phi_i)} \subseteq (\phi_i)^{|\circ|}$, as it is a change from a strict inequality to one that is not strict. It therefore suffices to show that $(\phi_i)^{|\circ|} \subseteq U$. We have

$$(\phi_i)^{|\circ|} = \left(\bigcup_{i=1}^{\infty} Y_i \right)^{|\circ|} = \left(\bigcup_{i=1}^{\infty} X_i \right)^{|\circ|} = \bigcap_{i=1}^{\infty} {X_i}^{|\circ|},$$

by Lemma 0.3.11, and we constructed the (X_i) so that ${X_i}^{|\circ|} \cap iB \subseteq U$. So for all $i \in \mathbb{N}$

$$\left(\bigcup_{j=1}^{\infty} {X_i}^{|\circ|} \right) \cap iB \subseteq {X_i}^{|\circ|} \cap iB \subseteq U.$$

Therefore

$$\begin{aligned}
(\phi_i)^{|\circ|} = \bigcap_{i=1}^{\infty} {X_i}^{|\circ|} = \left(\bigcap_{i=1}^{\infty} {X_i}^{|\circ|} \right) \cap E = \left(\bigcap_{i=1}^{\infty} {X_i}^{|\circ|} \right) \cap \left(\bigcup_{j=1}^{\infty} jB \right) \\
= \bigcup_{j=1}^{\infty} \left(\bigcap_{i=1}^{\infty} {X_i}^{|\circ|} \right) \cap jB.
\end{aligned}$$

As each part of the big union is a subset of U, the union is too, so we have shown $(\phi_i)^{|\circ|} \subseteq U$. □

Proposition 3.2.9. *If $(E, \mathcal{T}, B)$ is a locally convex space with compact barrel B, then $\mathcal{T}_b$ is locally convex and $(E, \mathcal{T}_b, B)$ is a Smith space.*

Proof. We prove this by showing that the filter base

$$\mathcal{N} = \{N_{(\phi_i)} \mid (\phi_i) \text{ converges to 0 in } (E^*, \|\text{-}\|_{B^{|\circ|}})\}$$

defines a locally convex topology $\mathcal{S}$ using [18, II.4.1 Proposition 1].[2] Then Proposition 3.2.8 implies $(E, \mathcal{T}_b)$ has the same neighbourhood filter at zero as $\mathcal{S}$, and therefore has the same neighbourhood filter at every point by Proposition 3.2.6 (iii).

We must therefore show that each $N_{(\phi_i)} \in \mathcal{N}$ is absorbent, absolutely convex and that $\alpha N_{(\phi_i)} \in \mathcal{N}$ for all $\alpha \in \mathbb{R}_{>0}$.

- Each $N_{(\phi_i)} \in \mathcal{N}$ is absorbent:

  We showed in Proposition 3.2.8 that $N_{(\phi_i)} \supseteq N_{(\phi_i)} \cap B = (\bigcap_{i=1}^{n} N_{\phi_i}) \cap B$ for some $n \in \mathbb{N}$. It is not hard to prove directly that N_{ϕ_i} is absorbent, but we can also deduce this from Lemma 0.1.4. By assumption, B is absorbent, so by Lemma 0.1.3 $\bigcap_{i=1}^{n} N_{\phi_i} \cap B$ is absorbent and so $N_{(\phi_i)}$ is absorbent.

- $N_{(\phi_i)}$ is absolutely convex:

  Let $\sum_{i \in I} \alpha_i x_i$ be a finite absolutely convex combination of elements of $N_{(\phi_i)}$. Then for all $i \in I$ and $j \in \mathbb{N}$, $|\phi_j(x_i)| < 1$. Therefore

  $$\left|\phi_j\left(\sum_{i \in I} \alpha_i x_i\right)\right| = \left|\sum_{i \in I} \alpha_i \phi_j(x_i)\right| \leq \sum_{i \in I} |\alpha_i| \cdot |\phi_j(x_i)| < \sum_{i \in I} |\alpha_i| \leq 1,$$

  so $\sum_{i \in I} \alpha_i x_i \in N_{(\phi_i)}$.

- $N_{(\phi_i)} \in \mathcal{N}$ implies $\alpha N_{(\phi_i)} \in \mathcal{N}$ for all $\alpha \in \mathbb{R}_{>0}$:

  We show this by proving that $\alpha N_{(\phi_i)} = N_{(\alpha^{-1}\phi_i)}$, analogously to Lemma 0.3.11 (ii), and then continuity of scalar multiplication on $(E^*, \|\text{-}\|_{B^\circ})$ shows that $(\alpha^{-1}\phi_i)_{i \in \mathbb{N}}$ converges to 0, so $N_{(\alpha^{-1}\phi_i)} \in \mathcal{N}$.

  $$\begin{aligned} x \in \alpha N_{(\phi_i)} &\Leftrightarrow \alpha^{-1} x \in N_{(\phi_i)} \Leftrightarrow \forall i \in \mathbb{N}. |\phi_i(\alpha^{-1} x)| < 1 \\ &\Leftrightarrow \forall i \in \mathbb{N}. |\alpha^{-1} \phi(x)| < 1 \Leftrightarrow x \in N_{(\alpha^{-1}\phi_i)}. \end{aligned}$$

  So the two sets are the same.

Therefore $\mathcal{T}_b$ is a locally convex topology. We know that $\mathcal{T}_b$ agrees with $\mathcal{T}$ on αB for all $\alpha \in \mathbb{R}_{>0}$, and that it is the finest such topology. Therefore it is the finest topology agreeing with $\mathcal{T}_b$ on αB for all $\alpha \in \mathbb{R}_{>0}$, so $(E, \mathcal{T}_b, B)$ is a Smith space, by our definition. □

[2] See [19, I.1.2 Proposition 2 and III.1.2 Proposition 1], then [18, I.1.5 Proposition 4] for the whole story.

Lemma 3.2.10. *Let $(E, \mathcal{T}, B)$ be a Smith space.*

(i) If $U \subseteq E$ is a set such that $U \cap \alpha B$ is $\mathcal{T}$-open in αB for all $\alpha \in \mathbb{R}_{>0}$, then U is $\mathcal{T}$-open.

(ii) If $C \subseteq E$ is a set such that $S \cap \alpha B$ is $\mathcal{T}$-closed in αB for all $\alpha \in \mathbb{R}_{>0}$, then C is $\mathcal{T}$-closed.

Proof.

(i) We can take $\mathcal{T}_b = \{U \subseteq E | \forall \alpha \in \mathbb{R}_{>0}. \exists U_\alpha \in \mathcal{T}. U \cap \alpha B = U\alpha \cap \alpha B\}$ as in Proposition 3.2.6. By part (ii) of that proposition, $\mathcal{T}_b$ is the finest topology agreeing with $\mathcal{T}$ on αB for all $\alpha \in \mathbb{R}_{>0}$, so by our assumption that $(E, \mathcal{T}, B)$ is Smith, $\mathcal{T}_b = \mathcal{T}$. Since $U \in \mathcal{T}_b$, we have that $U \in \mathcal{T}$.

(ii) We deduce this from part (i) as follows. We know that $C \cap \alpha B = C_\alpha \cap \alpha B$ for all $\alpha \in \mathbb{R}_{>0}$, where C_α is a $\mathcal{T}$-closed set. Now

$$\begin{aligned}(E \setminus C) \cap \alpha B = \alpha B \setminus C = \alpha B \setminus (\alpha B \cap C) = \alpha B \setminus (\alpha B \cap C_\alpha) \\ = (E \setminus C_\alpha) \cap \alpha B.\end{aligned}$$

Sine $E \setminus C_\alpha$ is $\mathcal{T}$-open for all $\alpha \in \mathbb{R}_{>0}$, we deduce from (i) that $E \setminus C$ is $\mathcal{T}$-open, and therefore C is $\mathcal{T}$-closed. □

We now introduce a notation. If $(E, \mathcal{T}, B)$ a locally convex space with compact barrel B, such as a Smith space, we define $(E, \mathcal{T}, B)^\beta$, or E^β where no confusion is possible, to be E^* with the topology defined by $\|\text{-}\|_{B^\circ}$. We use the letter β becaues it is associated to strong topologies, although we have not yet shown that for a Smith space $(E, \mathcal{T}, B)$, E^β is the dual space with the strong dual topology. The choice of letter from the relationship between the strong dual topology and uniform convergence on bounded sets. We have already shown in Lemma 3.2.2 that this is always a normed space. We aim to show that in the case of a Smith space, this is in fact a Banach space. To do this we first need a lemma.

Lemma 3.2.11. *Let $(E, \mathcal{T}, B)$ be a locally convex space with a compact barrel. The family*

$$\mathfrak{S}_B = \{S \subseteq \alpha B \mid \alpha \in \mathbb{R}_{>0}\}$$

is a saturated family (in the sense of [118, p. 81]) that covers E and consists of sets that are bounded.

Proof. There are three conditions to check for $\mathfrak{S}_B$ to be a saturated family ([118, p.81]).

(i) $\mathfrak{S}_B$ contains all subsets of its elements: Trivially implied by the definition.

(ii) $\mathfrak{S}_B$ contains all scalar multiples of its elements: If $S \subseteq \alpha B$, then $\beta S \subseteq \alpha\beta B$, so $\beta S \in \mathfrak{S}_B$.

(iii) $\mathfrak{S}_B$ contains the closed absolutely convex hull of all finite unions of its elements: Let $(S_i)_{i\in I}$ be a finite family of elements of $\mathfrak{S}_B$, with $(\alpha_i)_{i\in I}$ being defined such that $S_i \subseteq \alpha_i B$. Let j be the index of $\max_{i\in I} \alpha_i$. Then for all $i \in I$, $S_i \subseteq \alpha_j B$, and so $\bigcup_{i\in I} S_i \subseteq \alpha_j B$. Then $\alpha_j B$ is closed and absolutely convex, so the closed absolutely convex hull of $\bigcup_{i\in I} S_i$ is also a subset of $\alpha_j B$, hence an element of $\mathfrak{S}_B$.

We see that αB covers E because B is absorbent. Each $S \in \mathfrak{S}_B$ is bounded because compact sets are bounded (Lemma 0.1.14) and any subset of a bounded set is bounded. □

For any saturated family of sets on a locally convex space E, we can define a locally convex topology on E^* by using the polars of sets from that family as a base for 0-neighbourhoods ([118, III.3.2 Corollary] with $F = \mathbb{R}$).

Lemma 3.2.12. *Given a locally convex space with compact barrel $(E, \mathcal{T}, B)$, the $\mathfrak{S}_B$-topology on E^* is the $\|\text{-}\|_{B^\circ}$ topology.*

Proof. Suppose $N \subseteq E$ is a $\mathfrak{S}_B$ 0-neighbourhood, *i.e.* there exists an $S \in \mathfrak{S}_B$ such that $S^\circ \subseteq N$. Then $S \subseteq \alpha B$ for some $\alpha \in \mathbb{R}_{>0}$, so by Lemma 0.3.11(ii) and (iii), $\alpha^{-1}B^\circ = (\alpha B)^\circ \subseteq S^\circ \subseteq N$, so N is a $\|\text{-}\|_{B^\circ}$ 0-neighbourhood.

In the other direction, let N be a $\|\text{-}\|_{B^\circ}$ 0-neighbourhood, *i.e.* there exists some $\alpha \in \mathbb{R}_{>0}$ such that $\alpha B^\circ \subseteq N$. Then by Lemma 0.3.11 (ii), $(\alpha^{-1}B)^\circ \subseteq N$, so N is a $\mathfrak{S}_B$ 0-neighbourhood. □

Proposition 3.2.13. *Let $(E, \mathcal{T}, B)$ be a locally convex space with B a compact barrel. Then $(E, \mathcal{T}_b, B)^\beta$ is isomorphic to the completion of $(E, \mathcal{T}, B)^\beta$. In particular, if $(E, \mathcal{T}, B)$ is a Smith space, E^β is a Banach space.*

Proof. We use Grothendieck's completeness theorem ([118, Theorem IV.6.2]). This states that E^* is complete in the $\mathfrak{S}_B$ topology iff every $\phi : E \to \mathbb{R}$ which is continuous when restricted to any $S \in \mathfrak{S}_B$ is continuous. We want to use this to show that $(E, \mathcal{T}, B)^\beta$ is complete whenever $(E, \mathcal{T}, B)$ is a Smith space.

Let $(E, \mathcal{T}, B)$ be a Smith space and suppose we have a linear map $\phi : E \to \mathbb{R}$, and for all $S \in \mathfrak{S}_B$, $\phi|_S$ is continuous, where E has the subspace topology from $\mathcal{T}$. Then *a fortiori* we have that $\phi|_{\alpha B}$ is continuous, and so if $V \subseteq \mathbb{R}$ is an open set, there exists $U_\alpha \in \mathbb{R}$ such that $f^{-1}(V) \cap \alpha B = U_\alpha \cap \alpha B$. By Lemma 3.2.10, $f^{-1}(V) \in \mathcal{T}$, so ϕ is continuous. We have therefore shown E^β is complete for any Smith space $(E, \mathcal{T}, B)$, and so if $(F, \mathcal{S}, C)$ is a locally convex space with compact barrel B, then $(F, \mathcal{S}_b, C)^\beta$ is complete by Proposition 3.2.9.

It remains to show that if $(E, \mathcal{T}, B)$ is a locally convex space with compact barrel B, that $(E, \mathcal{T}_b, B)^\beta$ is isomorphic the completion of $(E, \mathcal{T}, B)$. By a standard theorem ([19, II.3.7 Proposition 13]) this can be proven by showing that $(E, \mathcal{T}, B)^\beta$ is dense in $(E, \mathcal{T}_b, B)^\beta$ and $(E, \mathcal{T}, B)^\beta$ has the subspace topology as a subset of $(E, \mathcal{T}_b, B)^\beta$ (it is a subset because $\mathcal{T}_b$ is finer than $\mathcal{T}$).

We show that $(E, \mathcal{T}, B)^\beta$ has the subspace topology as follows. We use $B^\circ_{\mathcal{T}}$ to mean the polar of B in $(E, \mathcal{T})^*$, and $B^\circ_{\mathcal{T}_b}$ to mean the polar of B in $(E, \mathcal{T}_b)^*$. The unit ball of $\|\text{-}\|_{B^\circ_{\mathcal{T}}}$ is $B^\circ_{\mathcal{T}}$. The unit ball of the subspace topology of $\|\text{-}\|_{B^\circ_{\mathcal{T}_b}}$

in $(E,\mathcal{T})^*$ is $B^{\circ}_{\mathcal{T}_b} \cap (E,\mathcal{T})^*$. By Lemma 0.3.13 these are equal, so the two norms and the topologies they generate are equal on $(E,\mathcal{T})^* = (E,\mathcal{T},B)^\beta$.

To show that $(E,\mathcal{T})^*$ is dense in $(E,\mathcal{T}_b)^*$, we use [118, IV.6.2 Corollary 1]. This states that if $(E, F, \langle -,- \rangle)$ is a duality and $\mathfrak{S}$ is a saturated family of weakly bounded sets covering E, and F_1 is the space of linear maps $\phi : E \to \mathbb{R}$ whose restrictions to each $S \in \mathfrak{S}$ are $\sigma(E, F)$-continuous, given the $\mathfrak{S}$-topology, then G_1 is complete and G, embedded in it via the pairing, is dense in it.

Now, $\mathfrak{S}_B$ is a saturated family of bounded sets covering E (Lemma 3.2.11) and every bounded set is weakly bounded because every weak 0-neighbourhood is a 0-neighbourhood in $\mathcal{T}$. We also have a duality $(E,(E,\mathcal{T})^*)$, by Proposition 0.3.1. Let $\phi : E \to \mathbb{R}$ be a linear map. By Lemma 3.2.7, ϕ is $\sigma(E,(E,\mathcal{T})^*)$-continuous when restricted to each $S \in \mathfrak{S}$ iff it is $\mathcal{T}$-continuous when restricted to each $S \in \mathfrak{S}$. So by the argument in the second paragraph, ϕ is $\mathcal{T}_b$ continuous. If ϕ is $\mathcal{T}_b$ continuous, it is also $\mathcal{T}$-continuous when restricted to any $S \in \mathfrak{S}$ (Proposition 3.2.6(ii)), so we have shown $(E,\mathcal{T}_b)^* = (E,\mathcal{T})^*_1$, so by [118, IV.6.2 Corollary 1], $(E,\mathcal{T})^*$ is dense in $(E,\mathcal{T}_b)^*$, and therefore $(E,\mathcal{T}_b,B)^\beta$ is the completion of $(E,\mathcal{T},B)^\beta$. □

We can now prove the following fact about bounded sets in Smith spaces.

Proposition 3.2.14. *In any Smith space $(E,\mathcal{T},B)$, if $S \subseteq E$ is weakly bounded, then there exists and $\alpha \in \mathbb{R}_{>0}$ such that $S \subseteq \alpha B$, and therefore S is bounded. Therefore $\{\alpha B\}_{\alpha \in \mathbb{R}_{>0}}$ is a fundamental family for both weakly bounded sets and bounded sets. We also have that for each compact set $C \subseteq E$ there is an $\alpha \in \mathbb{R}_{>0}$ such that $C \subseteq \alpha B$.*

Proof. Let $S \subseteq E$ be a weakly bounded (*i.e.* $\sigma(E,E^*)$-bounded) set. We first show that $S^{|\circ|}$ is a barrel, *i.e.* a set that is closed, absolutely convex and absorbent.

By Lemma 0.3.5, $S^{|\circ|}$ is absolutely convex and $\sigma(E^*,E)$-closed. The topology $\sigma(E^*,E)$ is, by definition, the coarsest locally convex topology such that for all $x \in E$, $\mathrm{ev}(x) : E^* \to \mathbb{R}$ is continuous. We proved in Lemma 3.2.4 that $\mathrm{ev}(x)$ was continuous for the $\|\text{-}\|_{B^\circ}$ norm for all $x \in E$, and therefore $\sigma(E^*,E)$ is coarser than the $\|\text{-}\|_{B^\circ}$ topology, so $S^{|\circ|}$ is also closed in this topology. To show that $S^{|\circ|}$ is absorbent, let $\phi \in E^*$. Since S is weakly bounded, there exists an $\alpha \in \mathbb{R}_{>0}$ such that $S \subseteq \alpha N_\phi$, where N_ϕ is the open $\sigma(E,E^*)$ 0-neighbourhood as in equation (2). So we have

$$\begin{aligned}
\forall x \in S. x \in \alpha N_\phi &\Leftrightarrow \forall x \in S. \alpha^{-1} x \in N_\phi \\
&\Leftrightarrow \forall x \in \alpha^{-1} S. x \in N_\phi \\
&\Leftrightarrow \forall x \in \alpha^{-1} S. |\phi(x)| < 1 \\
&\Rightarrow \forall x \in \alpha^{-1} S. |\phi(x)| \leq 1 \\
&\Leftrightarrow \phi \in (\alpha^{-1} S)^{|\circ|} \\
&\Leftrightarrow \phi \in \alpha S^{|\circ|}.
\end{aligned}$$

We have now shown $S^{|\circ|}$ is a barrel.

Since E^β is a Banach space, it is barrelled by [118, II.7.1 Corollary], and so $S^{|\circ|}$ is a 0-neighbourhood. Therefore there exists an $\alpha \in \mathbb{R}_{>0}$ such that $\alpha B^{|\circ|} \subseteq S^{|\circ|}$ (recall that $B^{|\circ|} = B^\circ$ by Lemma 0.3.6 and S° is the closed unit ball by Lemma 3.2.2). By Lemma 0.3.11

$$S^{|\circ||\circ|} \subseteq \left(\alpha B^{|\circ|}\right)^{|\circ|} = \alpha^{-1} B^{|\circ||\circ|}.$$

We then have

$$S \subseteq S^{|\circ||\circ|} \subseteq \alpha^{-1} B^{|\circ||\circ|} = \alpha^{-1} B$$

by the absolute bipolar theorem (Corollary 0.3.12).

Since αB is compact, it is $\mathcal{T}$-bounded, by Lemma 0.1.14. Therefore every weakly bounded set is $\mathcal{T}$-bounded, the other implication holding by definition. Therefore a set $S \subseteq E$ is (weakly) bounded iff it is a subset of some αB for $\alpha \in \mathbb{R}_{>0}$. Since compact sets are bounded, we also have that every compact set is contained in some αB. □

The preceding proof shows that $\mathfrak{S}_B$ is actually the family of weakly bounded sets in E, so E^β is in fact the strong dual, as defined in [118, IV.5], justifying our choice of Greek letter. We also see that the previous result and Lemma 3.2.10 imply that every Smith space is compactly generated (see [83, VII.8] and [75, page 230] for compactly generated spaces, and [2, Proposition 4.7] for Akbarov's proof of this).

We can prove continuity of maps from Smith spaces to topological vector spaces more easily using the following proposition.

Proposition 3.2.15. *Let $(E, \mathcal{T}, B)$ be a Smith space, $(F, \mathcal{S})$ a topological vector space, and $f : E \to F$ a linear map such that $f|_B$ is continuous. Then f is continuous.*

Proof. As $f|_B$ is continuous, we have that for all $V \in \mathcal{S}$, there exists a $U \in \mathcal{T}$ such that $f^{-1}(V) \cap B = U \cap B$. The first step in proving f is continuous is to prove that $f|_{\alpha B}$ is continuous for all $\alpha \in \mathbb{R}_{>0}$. So let $V \in \mathcal{S}$. Since $(F, \mathcal{S})$ is a topological vector space, $\alpha^{-1} V \in \mathcal{S}$. Therefore there is a $U_0 \in \mathcal{T}$ such that $f^{-1}(\alpha^{-1} V) \cap B = U_0 \cap B$. So

$$\begin{aligned} &\alpha(f^{-1}(\alpha^{-1}V) \cap B) = \alpha(U_0 \cap B) \\ &\Leftrightarrow f^{-1}(V) \cap \alpha B = \alpha U_0 \cap \alpha B. \end{aligned}$$

Since $\alpha U_0 \in \mathcal{T}$, we have shown $f|_{\alpha B}$ is continuous.

If we fix an open set $V \in \mathcal{S}$, and denote by U_α an open set such that $f^{-1}(V) \cap \alpha B = U_\alpha \cap \alpha B$, which we proved to exist in the previous paragraph for each $\alpha \in \mathbb{R}_{>0}$, we can see that $f^{-1}(V) \in \mathcal{T}$ by Lemma 3.2.10. □

Corollary 3.2.16. *Let $(E, \mathcal{T}, B)$ be a Smith space, $(F, \mathcal{S}, C)$ a locally convex space with compact barrel C, and $f : E \to F$ a linear map such that there is an $\alpha \in \mathbb{R}_{>0}$ such that $f(B) \subseteq \alpha C$ and $f|_B$ is continuous for $\mathcal{S}$. Then f is continuous from $(E, \mathcal{T})$ to $(F, \mathcal{S}_b)$.*

Proof. We first show that $f|_B$ is continuous from $(B, \mathcal{T}|_B)$ to $(F, \mathcal{S}_b)$. Let $V \in \mathcal{S}_b$. This means, in particular, that there exists a set $V_\alpha \in \mathcal{S}$ such that $V_\alpha \cap \alpha C = V \cap \alpha C$. Then

$$\begin{aligned}
&f^{-1}(V_\alpha \cap \alpha C) = f^{-1}(V \cap \alpha C) \\
&\Leftrightarrow f^{-1}(V_\alpha) \cap f^{-1}(\alpha C) = f^{-1}(V) \cap f^{-1}(\alpha C) \\
&\Leftrightarrow f^{-1}(V_\alpha) \cap B = f^{-1}(V) \cap B,
\end{aligned}$$

by the assumption that $B \subseteq f^{-1}(\alpha C)$. By the assumed continuity of $f|_B$ for $\mathcal{S}$, $f^{-1}(V_\alpha) \cap B = U \cap B$ for some $U \in \mathcal{T}$, so this shows that $f^{-1}(V) \cap B = U \cap B$ and therefore $f|_B$ is continuous for $\mathcal{S}_b$ as well. By Proposition 3.2.15, f is continuous from $(E, \mathcal{T})$ to $(F, \mathcal{S}_b)$. □

If E is a normed space, by the Banach-Alaoglu theorem [32, V.4 Theorem 2][3] Ball(E^*), of the dual norm, is compact in the weak-* (or $\sigma(E^*, E)$) topology. Since Ball(E^*) is absolutely convex and absorbent, $(E, \sigma(E^*, E)_b, B)$ is a Smith space (Proposition 3.2.9). We denote this by E^σ. This is known as the *bounded weak-* topology* [32, Definition V.5.3, Corollary V.5.5] [8, Chapter 1, Theorem 2.2], usually restricted to the case that E is Banach. We use the letter σ as it is associated to weak topologies (probably from *schwach*).

We now consider the embedding in the double dual, in particular

$$\begin{aligned}
\mathrm{ev} : E \to E^{\beta\sigma} \\
\mathrm{ev}(x)(\phi) = \phi(x),
\end{aligned}$$

where $(E, \mathcal{T}, B)$ is a Smith space.

Proposition 3.2.17. *If $(E, \mathcal{T}, B)$ is a Smith space, the map $\mathrm{ev} : E \to E^{\beta\sigma}$ is a linear homeomorphism preserving the unit ball. Therefore every Smith space is isomorphic to the bounded weak-* dual of a Banach space, which can be taken to be E^β.*

Proof. We first show that $x \in E$ implies $\mathrm{ev}(x) \in E^{\beta\sigma}$. The underlying space of $E^{\beta\sigma}$ is $(E^*, \|\text{-}\|_{B^\circ})^*$, so $\mathrm{ev}(x) \in E^{\beta\sigma}$ iff $\mathrm{ev}(x) : E^* \to \mathbb{R}$ is continuous with respect to $\|\text{-}\|_{B^\circ}$. This follows from Lemma 3.2.4.

To show that ev is continuous, we first show it is continuous if $E^{\beta\sigma}$ is given the $\sigma(E^{\beta\sigma}, E^\beta)$ topology. A subbasis for open neighbourhoods is given by the family of sets

$$N_\phi = \{\Phi \in E^{\beta\sigma} \mid |\Phi(\phi)| < 1\}$$

where $\phi \in E^\beta$ (see (2)). Because preimages preserve intersections, we only need to show that $\mathrm{ev}^{-1}(N_\phi)$ is open for all $\phi \in E^\beta$. So

$$\begin{aligned}
\mathrm{ev}^{-1}(N_\phi) &= \{x \in E \mid \mathrm{ev}(x) \in N_\phi\} = \{x \in E \mid |\mathrm{ev}(x)(\phi)| < \epsilon\} \\
&= \{x \in E \mid |\phi(x)| < \epsilon\} = \{x \in E \mid \phi(x) \in (-\epsilon, \epsilon)\} = \phi^{-1}((-\epsilon, \epsilon)),
\end{aligned}$$

[3] Though the theorem is stated for Banach spaces in this reference, the proof does not use completeness.

which is open because ϕ is continuous.

We denote the unit ball of $E^{\beta\sigma}$ by C. This is the polar of the unit ball of E^{β}, which is B^{o} (Lemma 3.2.2), but the polars are with respect to different pairings. We show that

$$B = \text{ev}^{-1}(C). \tag{3.4}$$

By definition

$$C = \{\Phi \in E^{\beta\sigma} \mid \forall \phi \in B^{\text{o}}.|\Phi(\phi)| \leq 1\},$$

so

$$\begin{aligned}\text{ev}^{-1}(C) &= \{x \in E \mid \text{ev}(x) \in C\} = \{x \in E \mid \forall \phi \in B^{\text{o}}.|\text{ev}(x)(\phi)| \leq 1\} \\ &= \{x \in E \mid \forall \phi \in B^{\text{o}}.|\phi(x)| \leq 1\} = B^{\text{oo}},\end{aligned}$$

and $B^{\text{oo}} = B$ by Corollary 0.3.10 as B is closed.

Now, let, U be an open subset of $E^{\beta\sigma}$, *i.e.* $U \in \sigma(E^{\beta\sigma}, E^{\beta})_b$, which is to say that for all $\alpha > 0$, there exists $U_\alpha \in \sigma(E^{\beta\sigma}, E^{\beta})$ such that $U \cap \alpha C = U_\alpha \cap \alpha C$. Then

$$\begin{aligned}\text{ev}^{-1}(U) \cap \alpha B &= \text{ev}^{-1}(U) \cap \alpha\text{ev}^{-1}(C) && (3.4) \\ &= \text{ev}^{-1}(U \cap \alpha C) && \text{linearity} \\ &= \text{ev}^{-1}(U_\alpha \cap \alpha C) \\ &= \text{ev}^{-1}(U_\alpha) \cap \alpha B.\end{aligned}$$

We already showed that $\text{ev}^{-1}(U_\alpha)$ is $\mathcal{T}$-open, so U is $\mathcal{T}$-open by Lemma 3.2.10, and we have shown that ev is continuous.

To see that ev is injective, suppose $x, y \in E$ and $\text{ev}(x) = \text{ev}(y)$. Then for all $\phi \in E^*$, we have $\text{ev}(x)(\phi) = \text{ev}(y)(\phi)$, *i.e.* $\phi(x) = \phi(y)$ and so $\phi(x - y) = 0$. Since the pairing between E and E^* is separating, $x = y$ (Proposition 0.3.1).

To show that ev is surjective, we first show that $\text{ev}(B) = C$. We do this by showing first that $\text{Ball}(E^{\beta}) = \text{ev}(B)^{\text{o}}$, where the polar is with respect to the $(E^{\beta}, E^{\beta\sigma})$ duality. We have

$$\text{ev}(B)^{\text{o}} = \{\phi \in E^{\beta} \mid \forall \Phi \in \text{ev}(B).|\Phi(\phi)| \leq 1\},$$

and $\Phi \in \text{ev}(B)$ iff there exists an $x \in B$ such that $\text{ev}(x) = \Phi$. Therefore

$$\text{ev}(B)^{\text{o}} = \{\phi \in E^{\beta} \mid \forall x \in B.|\text{ev}(x)(\phi)| \leq 1\} = \{\phi \in E^{\beta} \mid \forall x \in B.|\phi(x)| \leq 1\}.$$

This is equal to $\text{Ball}(E^{\beta}) = B^{\text{o}}$ (the polar being with respect to the (E, E^{β}) duality this time).

Taking polars, we get that $C = \text{Ball}(E^{\beta})^{\text{o}} = \text{ev}(B)^{\text{oo}}$. Since ev is continuous, $\text{ev}(B)$ is a compact, hence closed, subset of $E^{\beta\sigma}$, and it is also absolutely convex by the linearity of ev. So $\text{ev}(B)^{\text{oo}} = \text{ev}(B)$, and we conclude that $C = \text{ev}(B)$. By linearity of ev we also obtain $\alpha C = \text{ev}(\alpha B)$ for all $\alpha \in \mathbb{R}_{>0}$.

Now, let $\Phi \in E^{\beta\sigma}$. Since C is absorbent, being the unit ball of a norm, there is an $\alpha \in \mathbb{R}_{>0}$ such that $\Phi \in \alpha C$. Since $\text{ev}(\alpha B) = \alpha C$, there is an $x \in \alpha B \subseteq E$ such that $\text{ev}(x) = \Phi$, so we have shown that ev is surjective.

Since ev is a continuous bijection, to show that it is a homeomorphism we only need to show that it is an open mapping. So let $U \in \mathcal{T}$. For all $\alpha \in \mathbb{R}_{>0}$, $\mathrm{ev}|_{\alpha B}$ is a continuous bijection of compact Hausdorff spaces from αB to αC, and therefore an open mapping, so $\mathrm{ev}(U \cap \alpha B)$ is relatively open in αC, so is equal to $V_\alpha \cap \alpha C$ for some V_α that is open in $E^{\beta\sigma}$. Therefore for all $\alpha \in \mathbb{R}_{>0}$

$$\begin{aligned} \mathrm{ev}(U) \cap \alpha C &= \mathrm{ev}(U) \cap \mathrm{ev}(\alpha B) \\ &= \mathrm{ev}(U \cap \alpha B) && \text{ev bijective} \\ &= V_\alpha \cap \alpha C, \end{aligned}$$

so by Lemma 3.2.10, $\mathrm{ev}(U)$ is open in $E^{\beta\sigma}$. □

The preceding proposition shows that our redefinition of Smith space agrees with Akbarov's [2, Theorem 4.11]. Results of the above nature go back to Dixmier's fundamental work [28, Théorème 19], where instead of a dealing with a topology on E one chose a subspace of the dual, and Ng's improvement of this result [96, Theorem 1].

Corollary 3.2.18. *For each Smith space $(E, \mathcal{T}, B)$, the normed space $(E, \|\text{-}\|_B)$ is complete. Therefore there are forgetful functors $U_1 : \mathbf{Smith}_1 \to \mathbf{Ban}_1$ and $U_\infty : \mathbf{Smith} \to \mathbf{Ban}$.*

Proof. By Proposition 3.2.17, ev is an isomorphism between E and $E^{\beta\sigma}$, and the unit ball B of E is mapped to the unit ball C of $E^{\beta\sigma}$. Therefore $\mathrm{ev} : (E, \|\text{-}\|_B) \to (E^{\beta\sigma}, \|\text{-}\|_C)$ is an isometry of normed spaces. Since $(E^{\beta\sigma}, \|\text{-}\|_C)$ is the dual of a Banach space, it is a Banach space [32, Corollary II.3.9], and therefore E is.

To show that U_1 exists, we only need to observe that a map $f : (E, \mathcal{T}, B) \to (F, \mathcal{S}, C)$ in $\mathbf{Smith}_1$ maps B into C and so is bounded of norm ≤ 1 by Lemma 0.1.8.

For U_∞, we need a different argument. Let $f : (E, \mathcal{T}, B) \to (F, \mathcal{S}, C)$ be a continuous map of Smith spaces. The set $f(B) \subseteq F$ is compact, and therefore there exists $\alpha \in \mathbb{R}_{>0}$ such that $f(B) \subseteq \alpha C$ (Proposition 3.2.14). Therefore f is bounded with norm $\leq \alpha$ (Lemma 0.1.8). □

Corollary 3.2.19. *If $f : (E, \mathcal{T}, B) \to (F, \mathcal{S}, C)$ is a continuous linear bijection of Smith spaces, it is an isomorphism,* i.e. *the inverse is continuous.*

Proof. By Corollary 3.2.18 f is a bounded surjective map of Banach spaces. Therefore it is an open mapping (*i.e.* the image of an open set is open) by the open mapping theorem [24, §III.12.1] [118, III.2.1 Corollary 1] [32, Theorem II.2.1]. The open unit ball of E contains zero, so 0 is in the $\|\text{-}\|_C$-interior of $f(B)$. Therefore there exists a $\beta \in \mathbb{R}_{>0}$ such that $\beta C \subseteq f(B)$.

To show that f^{-1} is continuous, it suffices to show that $f^{-1}|_C$ is continuous. First, observe that as $\beta C \subseteq f(B)$, $C \subseteq \beta^{-1} f(B) = f(\beta^{-1} B)$. Now, $f|_{\beta^{-1}B} : \beta^{-1} B \to f(\beta^{-1} B)$ is a continuous bijection of compact Hausdorff spaces, and therefore a homeomorphism. So for any open set $U \subseteq E$, there exists an open set $V \subseteq F$ such that $f|_{\beta^{-1}B}(U \cap \beta^{-1}B) = V \cap f(\beta^{-1}B)$. As f is a bijection, $f(U \cap \beta^{-1}B) = f(U) \cap f(\beta^{-1}B)$, so we have

$$f(U) \cap f(\beta^{-1}B) = V \cap f(\beta^{-1}B),$$

and therefore $f(U) \cap C = V \cap C$, so $f(U) \cap C = (f^{-1}|_C)^{-1}(U)$ is relatively open in C. This shows that $f^{-1}|_C$ is continuous, so f^{-1} is continuous by Proposition 3.2.15. □

3.2.1 β and σ as functors

We now show how to define $\text{-}^{\beta} : \mathbf{Smith} \to \mathbf{Ban}^{\mathrm{op}}$ and $\text{-}^{\sigma} : \mathbf{Normed}^{\mathrm{op}} \to \mathbf{Smith}$, extending their definition on objects.

Let $f : (E, \mathcal{T}, B) \to (F, \mathcal{S}, C)$ be a continuous linear map of Smith spaces. Define

$$f^{\beta}(\psi) = \psi \circ f, \tag{3.5}$$

where $\psi \in F^{\beta}$.

Proposition 3.2.20. *The above definition of f^{β} defines a functor $\text{-}^{\beta} : \mathbf{Smith} \to \mathbf{Ban}^{\mathrm{op}}$ and $\mathbf{Smith}_1 \to \mathbf{Ban}_1^{\mathrm{op}}$.*

Proof. If $\psi \in F^{\beta}$, then as it is the composite of two continous linear maps, $\psi \circ f$ is continuous and linear, so is an element of E^{β}.

If $\alpha\phi + \beta\psi$ is a linear combination in F^{β}, then for all $x \in E$

$$\begin{aligned} f^{\beta}(\alpha\phi + \beta\psi)(x) &= (\alpha\phi + \beta\psi)(x) = \alpha\phi(x) + \beta\psi(x) = \alpha f^{\beta}(\phi)(x) + \beta f^{\beta}(\psi)(x) \\ &= (\alpha f^{\beta}(\phi) + \beta f^{\beta}(\psi))(x), \end{aligned}$$

so f^{β} is a linear map.

Since f is continuous, $f(B) \subseteq F$ is compact, so there exists an $\alpha \in \mathbb{R}_{>0}$ such that $f(B) \subseteq \alpha C$ by Proposition 3.2.14 (in the case that $f \in \mathbf{Smith}_1(E, F)$ we already know $f(B) \subseteq C$ so do not need to prove this).

We show that $f^{\beta}(C^{\mathrm{o}}) \subseteq \alpha B^{\mathrm{o}}$. If $\psi \in C^{\mathrm{o}}$ we have that for all $x \in C.\psi(x) \leq 1$. Since $f(B) \subseteq \alpha C$, we have $f(\alpha^{-1}B) \subseteq C$, by linearity. Taking these two facts together, we have that

$$\begin{aligned} \forall x \in \alpha^{-1}B.\psi(f(x)) \leq 1 &\Rightarrow \forall x \in \alpha^{-1}B.f^{\beta}(\psi)(x) \leq 1 \\ &\Leftrightarrow f^{\beta}(\psi) \in \alpha^{-1}B^{\mathrm{o}} = \alpha B^{\mathrm{o}}, \end{aligned}$$

by Lemma 0.3.11 (ii). We then use Lemma 0.1.8 to deduce that $\|f^{\beta}\| \leq \alpha$, so f^{β} is bounded, and therefore a morphism in $\mathbf{Ban}$ from $F^{\beta} \to E^{\beta}$. If $f \in \mathbf{Smith}_1(E, F)$, then previous argument shows $f^{\sigma}(C^{\mathrm{o}}) \subseteq B^{\mathrm{o}}$ so $\|f^{\beta}\| \leq 1$ and $f^{\beta} \in \mathbf{Ban}_1(F^{\beta}, E^{\beta})$.

Let id_E be an identity map of Smith spaces. Then if $\phi \in E^{\beta}$, $\mathrm{id}_E^{\beta}(\phi) = \phi \circ \mathrm{id}_E = \phi$, so $\mathrm{id}_E^{\beta} = \mathrm{id}_{E^{\beta}}$. If $f : E \to F$ and $g : F \to G$ are maps of Smith spaces and $\psi \in G^{\beta}$

$$(g \circ f)^{\beta}(\psi) = \psi \circ g \circ f = f^{\beta}(\psi \circ g) = (f^{\beta} \circ g^{\beta})(\psi),$$

which finishes the proof that -^{β} is a contravariant functor. □

Now let $f : E \to F$ be a bounded (or, equivalently, continuous) map of normed spaces. Define

$$f^\sigma(\psi) = \psi \circ f, \tag{3.6}$$

where $\psi \in F^\sigma$.

Proposition 3.2.21. *The definition of f^σ given above defines a functor -^σ : $\mathbf{Normed}^{\mathrm{op}} \to \mathbf{Smith}$ and $\mathbf{Normed}_1^{\mathrm{op}} \to \mathbf{Smith}_1$.*

Proof. Since linearity and continuity of functions are preserved under composition, we have that $\psi \circ f$ is always an element of E^σ for any $f \in \mathbf{Normed}(E, F)$ and $\psi \in F^\sigma$. The proof that f^σ is linear is identical to the proof of the linearity of f^β in Proposition 3.2.20, so is omitted.

To show f^σ is continuous from the topology $\sigma(F^\sigma, F)_b$ to $\sigma(E^\sigma, E)_b$, we first show that it is continuous from $\sigma(F^\sigma, F)$ to $\sigma(E^\sigma, E)$. We use the neighbourhood definition of continuity. Let N_x, where $x \in E$, be a subbasic neighbourhood in F for the $\sigma(E^\sigma, E)$. We can show that $N_{f(x)} \subseteq (f^\sigma)^{-1}(N_x)$ as follows:

$$\begin{aligned}\psi \in N_{f(x)} &\Leftrightarrow |\psi(f(x))| < 1 \Leftrightarrow |(\psi \circ f)(x)| < 1 \Leftrightarrow f^\sigma(\psi) \in N_x \\ &\Leftrightarrow \psi \in (f^\sigma)^{-1}(N_x).\end{aligned}$$

Since preimages preserve intersections, we have that the preimage of every basic 0-neighbourhood in the $\sigma(E^\sigma, E)$-topology is a 0-neighbourhood in the $\sigma(F^\sigma, F)$-topology, establishing continuity with respect to these topologies.

To show continuity for the corresponding bounded weak-* topologies, we can first see that, as $\sigma(F^\sigma, F)$ is coarser than $\sigma(F^\sigma, F)_b$, f^σ is continuous from $(F, \sigma(F^\sigma, F)_b)$ to $(E, \sigma(E^\sigma, E))$. We therefore know that $f^\sigma|_{C^\circ}$ is continuous with the same topologies. Now, since f is bounded, we can apply the same argument used in Proposition 3.2.20 to deduce $f^\beta(C^\circ) \subseteq \alpha B^\circ$ from $f(B) \subseteq \alpha C$ to f^σ instead and deduce that $f^\sigma(C^\circ) \subseteq \alpha B^\circ$, with $\alpha \leq 1$ n the case that $f \in \mathbf{Normed}_1(E, F)$. We then apply Corollary 3.2.16 to deduce that f^σ is continuous from $(F, \sigma(F^\sigma, F)_b)$ to $(E, \sigma(E^\sigma, E)_b)$. In the case that $f \in \mathbf{Normed}_1(E, F)$ this also shows $f^\sigma \in \mathbf{Smith}_1(F^\sigma, E^\sigma)$.

The proof of preservation of identity maps and composition of maps is similar to that in 3.2.20 and so is omitted. □

We now define $\eta_E : E \to E^{\beta\sigma}$ in **Smith** and $\epsilon_E : E \to E^{\sigma\beta}$ in **Normed** as

$$\eta_E(x)(\phi) = \phi(x) \qquad \text{for } x \in E \text{ and } \phi \in E^\beta \tag{3.7}$$

$$\epsilon_E(x)(\phi) = \phi(x) \qquad \text{for } x \in E \text{ and } \phi \in E^\sigma \tag{3.8}$$

Theorem 3.2.22. *The families of maps η_E and ϵ_E define the unit and counit of an adjunction $\beta \dashv \sigma$, for* **Normed** *and* **Smith** *and also* $\mathbf{Normed}_1$ *and* $\mathbf{Smith}_1$*. The map η_E is an isomorphism, while ϵ_E is an isomorphism iff E is complete.*

Proof. First, observe that the definition of η_E coincides with that of ev in Proposition 3.2.17 and that ev is proven there to be defined with the correct codomain and to be an isomorphism in **Smith**. In the course of the proof it is shown that for $(E, \mathcal{T}, B)$ a Smith space, $\mathrm{ev}(B)$ is the unit ball of $E^{\beta\sigma}$, there called C, so it is also an isomorphism in $\mathbf{Smith}_1$.

We therefore move on to proving that η_E is natural. Let $f \in \mathbf{Smith}(E, F)$. We want to show that

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \eta_E}\downarrow & & \downarrow{\scriptstyle \eta_F} \\ E^{\beta\sigma} & \xrightarrow[f^{\beta\sigma}]{} & F^{\beta\sigma} \end{array}$$

commutes, which is to say, that if $x \in E$ and $\psi \in F^\beta$, then

$$\eta_F(f(x))(\psi) = f^{\beta\sigma}(\eta_E(x))(\psi)$$

For the left hand side, we have $\eta_F(f(x))(\psi) = \psi(f(x))$. For the right hand side, we have

$$f^{\beta\sigma}(\eta_E(x))(\psi) = \eta_E(x)(f^\beta(\psi)) = f^\beta(\psi)(x) = \psi(f(x)).$$

Therefore the diagram commutes, and we have shown η is natural.

We now move on to showing that ϵ_E is defined correctly and is natural. The map ϵ_E has the same definition as $\langle x, \text{-}\rangle$ for the pairing between the space E and its continuous dual E^* (Proposition 0.3.1). Therefore $\epsilon_E(x)$ is linear and continuous for the $\sigma(E^\sigma, E)$-topology on E^σ. Since $\sigma(E^\sigma, E)_b$ is finer than $\sigma(E^\sigma, E)$, $\epsilon_E(x)$ is also continuous in that topology, and therefore is an element of $E^{\sigma\beta}$. We also have that ϵ_E is linear by using Proposition 0.3.1 again.

We show that ϵ_E is bounded as follows. Let B be the unit ball of E, $C = B^{\mathrm{o}}$ the unit ball of E^σ, and $D = C^{\mathrm{o}}$ the unit ball of $E^{\sigma\beta}$. We want to show that $\epsilon_E(B) \subseteq D$. So let $x \in B$. By Corollary 0.3.10, $B = B^{\mathrm{oo}} = C^{\mathrm{o}}$. Then

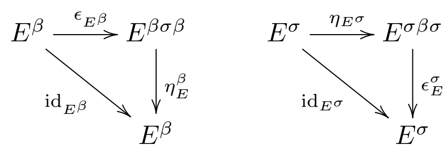
$$\forall \phi \in C. |\phi(x)| \leq 1 \Leftrightarrow \forall \phi \in C. |\epsilon_E(x)(\phi)| \leq 1 \Leftrightarrow \epsilon_E(x) \in C^{\mathrm{o}} = D.$$

So ϵ_E is bounded with norm ≤ 1, therefore a map in $\mathbf{Normed}_1$. If ϵ_E is bijective, the above argument also shows its inverse has norm ≤ 1, so it would be an isomorphism in $\mathbf{Normed}_1$.

We show that ϵ_E is bijective, and therefore an isomorphism in $\mathbf{Normed}_1$, iff E is a Banach space. In Proposition 3.2.13 we have seen that $E^{\sigma\beta}$ is the completion of $(E^\sigma, \sigma(E^\sigma, E), B^{\mathrm{o}})^\beta$, under the inclusion mapping. By Proposition 0.3.2, ϵ_E maps E bijectively onto $(E^\sigma, \sigma(E^\sigma, E))^*$, and we showed in the previous paragraph that this mapping preserves the norm. Therefore ϵ_E shows that $E^{\sigma\beta}$ is a completion of E. This is an isomorphism iff E is already complete, *i.e.* a Banach space.

The proof that ϵ is natural is similar to the proof that η is natural, so is omitted.

We now show that the following diagrams commute, which are the unit-counit diagrams for showing that $\beta \dashv \sigma$ (Theorem 0.4.1 (v)).



Note that E is a Smith space in the left triangle, while the triangle itself is in **Normed**, and so is reversed from its usual appearance. In the triangle on the right, E is a normed space and the triangle is in **Smith**.

To show the left triangle commutes, let $\phi \in E^\beta$ and $x \in E$. Then

$$\eta_E^\beta(\epsilon_{E^\beta}(\phi))(x) = \epsilon_{E^\beta}(\phi)(\eta_E(x)) = \eta_E(x)(\phi) = \phi(x).$$

As this holds for all $x \in X$ and $\phi \in E^\beta$, we get $\eta_E^\beta \circ \epsilon_{E^\beta}(\phi) = \mathrm{id}_{E^\beta}$ as required.

The proof that the right triangle commutes is similar, with σ replacing β and the rôles of η and ϵ reversed, so is omitted.

During the proof that η and ϵ are well defined, we already showed that η is always an isomorphism and ϵ_E an isomorphism whenever E is Banach. □

The following corollary is immediate.

Corollary 3.2.23. *The functors -$^\beta$ and -$^\sigma$ define equivalences*

$$\mathbf{Ban}^{\mathrm{op}} \simeq \mathbf{Smith}$$
$$\mathbf{Ban}_1^{\mathrm{op}} \simeq \mathbf{Smith}_1.$$

We also have

Corollary 3.2.24. *If $(E, \mathcal{T}, B)$ and $(F, \mathcal{S}, C)$ are Smith spaces, then a linear map $f : E \to F$ is continuous from $\mathcal{T}$ to $\mathcal{S}$ iff it is continuous from $\sigma(E, E^\beta)$ to $\sigma(F, F^\beta)$. In particular, if E_*, F_* are Banach spaces, then a linear map $E_*^\sigma \to F_*^\sigma$ is continuous on the Smith space topologies iff it is weak-* continuous.*

Proof. Suppose $f : E \to F$ is continuous from $\sigma(E, E^\beta) \to \sigma(F, F^\beta)$. By Lemma 3.2.7, B is $\sigma(E, E^\beta)$-compact, so $f(B)$ is $\sigma(F, F^\beta)$-compact, and therefore $\mathcal{S}$-compact because $\mathcal{S}$ is a finer topology. By Proposition 3.2.14, $f(B) \subseteq \alpha C$ for some $\alpha \in \mathbb{R}_{>0}$, and so by Corollary 3.2.16, f is continuous from $\mathcal{T}$ to $\mathcal{S}$.

If, on the other hand, we start with $f : E \to F$ being continuous from $\mathcal{T}$ to $\mathcal{S}$, we have that $f^\beta : F^\beta \to E^\beta$. If we consider the usual pairings between the spaces E, F and their duals E^β, F^β, we have, for all $\phi \in F^\beta$ and $x \in E$

$$\langle f^\beta(\phi), x\rangle = \langle \phi \circ f, x\rangle = \phi(f(x)) = \langle \phi, f(x)\rangle,$$

so by Proposition 0.3.3 f is continuous from $\sigma(E, E^\beta)$ to $\sigma(F, F^\beta)$.

The statement for E_* and F_* follows from the fact that the counit map $\epsilon_{E_*} : E_* \to E_*^{\sigma\beta}$ is an isometry of Banach spaces (Theorem 3.2.22. □

We prove one more fact that we will need later.

Proposition 3.2.25. *Let $f : E \to F$ be a bounded map of Banach spaces.*

(i) If $f(E)$ is dense in F, then f^σ is injective.

(ii) If f is injective, $f^\sigma(F^\sigma)$ is dense in E^σ.

Proof.

(i) Let $\phi, \psi \in F^\sigma$ such that $f^\sigma(\phi) = f^\sigma(\psi)$. This means ϕ and ψ agree on all elements of $f(E)$, a dense subset of F. As they are continuous, $\phi = \psi$.

(ii) The set $f^\sigma(F^\sigma)$ is a subspace of E^σ, so is a convex set. Therefore its closure in the Smith topology of E^σ equals its closure in $\sigma(E^\sigma, E^{\sigma\beta}) = \sigma(E^\sigma, E)$ (Proposition 0.3.4), which equals its bipolar by Corollary 0.3.10. So

$$\begin{aligned} \mathrm{cl}\left(f^\sigma(F^\sigma)\right) &= f^\sigma(F^\sigma)^{\mathrm{oo}} \\ &= f^{-1}(F^{\sigma\mathrm{o}})^{\mathrm{o}} && \text{Lemma 0.3.14} \\ &= f^{-1}(\{0\})^{\mathrm{o}} \\ &= \{0\}^{\mathrm{o}} && f \text{ injective} \\ &= F^\sigma. \end{aligned}$$

□

3.3 Compact Convex Sets and Smith Base-Norm Spaces

We first define Smith base-norm spaces. A *Smith base-norm space* is a quadruple $(E, \mathcal{T}, E_+, \tau)$, where $(E, \mathcal{T})$ is a locally convex topology, E_+ is a closed positive cone in this topology, and τ is a $\mathcal{T}$-continuous map $E \to \mathbb{R}$ such that (E, E_+, τ) is a base-norm space, and $(E, \mathcal{T}, \mathrm{absco}(B_E))$ is a Smith space. A trace-preserving morphism $f : (E, \mathcal{T}, E_+, \tau) \to (F, \mathcal{S}, F_+, \sigma)$ of Smith base-norm spaces is a continuous linear map that is a trace-preserving morphism of the underlying base-norm spaces. Trace-reducing maps are defined in a similar manner and Smith spaces with each kind of map form the categories **SBNS** and $\mathbf{SBNS}_{\leq 1}$, respectively.

In the following, we will often need to consider, given a topological vector space E, the map $c : \mathbb{R} \times E \times E \to E$ defined by

$$c(\alpha, x, y) = \alpha x + (1 - \alpha) y. \tag{3.9}$$

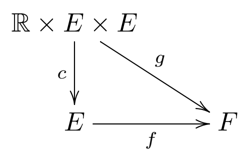
This mapping can be written as

$$c = + \circ ((\text{-} \cdot \text{-}) \times (\text{-} \cdot \text{-})) \circ (\mathrm{id}_R \times \sigma_{\mathbb{R},E} \times \mathrm{id}_E) \circ (\langle \mathrm{id}_{\mathbb{R}}, 1 - \text{-} \rangle \times \mathrm{id}_{E \times E}),$$

which is therefore a continuous map by the definition of a topological vector space.

Lemma 3.3.1. *Let $(E, \mathcal{T})$ be a topological vector space, and $X \subseteq E$ a compact convex subset. Let $B = \operatorname{absco}(X)$ and $f : (E, \mathcal{T}) \to (F, \mathcal{S})$ be a linear map such that $f|_X$ is continuous. Then $f|_B$ is continuous.*

Proof. We first define $g : \mathbb{R} \times E \times E$ such that



commutes.

Define $g(\alpha, x, y) = \alpha f(x) + (1-\alpha) f(y)$. We see that $f(c(\alpha, x, y)) = f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha) f(y) = g(\alpha, x, y)$ by linearity of f. We use Corollary A.2.2 to prove that $g|_{[0,1]\times X\times X}$ is continuous. Let $(\alpha_i, x_i, y_i)_{i\in I}$ be a net converging in the product topology to (α, x, y), everything being contained in $[0, 1] \times X \times X$. Then

$$\begin{aligned}
&\lim_{i\in I} g(\alpha_i, x_i, y_i) \\
&= \lim_{i\in I} \alpha_i f(x_i) + (1-\alpha_i) f(y_i) \\
&= \lim_{i\in I} \alpha_i f(x_i) + \lim_{i\in I} (1-\alpha_i) f(y_i) && + \text{ continuous} \\
&= \left(\lim_{i\in I} \alpha_i\right) \cdot \left(\lim_{i\in I} f(x_i)\right) + \left(\lim_{i\in I} (1-\alpha_i)\right) \cdot \left(\lim_{i\in I} f(y_i)\right) && \cdot \text{ continuous} \\
&= \alpha f(x) + \left(\lim_{i\in I} (1-\alpha_i)\right) f(y) && f|_X \text{ continuous} \\
&= \alpha f(x) + (1-\alpha) f(y) && +, - \text{ continuous} \\
&= g(\alpha, x, y),
\end{aligned}$$

which establishes the continuity.

To show that $f|_B$ is continuous, we first show that for each closed set $C \subseteq F$, $f^{-1}(C) \cap B$ is closed. We know that $g^{-1}(C) \cap [0, 1] \times X \times X$ is closed, by the continuity of $g|_{[0,1]\times X\times X}$. As $g = f \circ c$, we have that $c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$ is closed, and as it is a closed subset of a compact space, it is compact. Therefore $c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$ is compact, and therefore closed. If we show $f^{-1}(C) \cap B = c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$, we will have shown it is closed. If $x' \in f^{-1}(C) \cap B$, then as $B = \operatorname{co}(-X \cup X)$, we have that there are $x, y \in X$ and $\alpha \in [0, 1]$ such that $\alpha x + (1-\alpha)y = x'$, *i.e.* $c(\alpha, x, y) = x'$. So $(\alpha, x, y) \in c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$, and $x' \in c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$. If, on the other hand, $x' \in c(c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X)$, there exists $(\alpha, x, y) \in c^{-1}(f^{-1}(C)) \cap [0, 1] \times X \times X$ such that $x' = c(\alpha, x, y)$, so in particular, $c(\alpha, x, y) \in f^{-1}(C)$ and $c(\alpha, x, y) = \alpha x + (1-\alpha)y \in \operatorname{co}(-X \cup X) = B$, so $x' \in f^{-1}(C) \cap B$.

Now, let $V \subseteq F$ be an open set, so $F \setminus V$ is closed. Then $f^{-1}(F \setminus V) \cap B$ is closed in E, and so $E \setminus (f^{-1}(F \setminus V) \cap B)$ is open in E. Now

$$\begin{aligned}(E \setminus (f^{-1}(F \setminus V) \cap B)) \cap B &= B \setminus (f^{-1}(F \setminus V) \cap B)\\ &= B \setminus ((E \setminus f^{-1}(V)) \cap B)\\ &= B \setminus (B \setminus f^{-1}(V))\\ &= f^{-1}(V) \cap B,\end{aligned}$$

so $f^{-1}(V) \cap B$ is relatively open in B, as required. □

Proposition 3.3.2. *If (E, E_+, τ) is a pre-base-norm space, $\mathcal{T}$ a locally convex topology on E in which B_E is compact, then $(E, \mathcal{T}_b, E_+, \tau)$ is a Smith base-norm space, where $\mathcal{T}_b$ is taken with respect to the compact barrel* $\mathrm{absco}(B_E)$.

Proof. If B_E is empty, then by Lemma 2.2.1 $E = 0$ and the result holds tautologically as there is only one topology on E, which is Smith. Therefore we now assume that $B_E \neq \emptyset$. As it is a product of compact sets, the set $[0,1] \times B_E \times B_E \subseteq \mathbb{R} \times E \times E$ is compact, where each E has $\mathcal{T}$ as its topology. Because B_E is already convex, $c([0,1] \times B_E \times B_E) = \mathrm{co}(-B_E \cup B_E)$, and this is $\mathrm{absco}(B_E)$ (Lemma 0.1.1). As it is the image of a compact set under a continuous map, we have shown $\mathrm{absco}(B_E)$ is compact in $\mathcal{T}$, and therefore radially compact, so (E, E_+, τ) is a base-norm space. The set $\mathrm{absco}(B_E)$ is also absolutely convex, and is absorbent by Lemma 2.2.3, so is a compact barrel. We can therefore define $\mathcal{T}_b$ with respect to it, and obtain a Smith space $(E, \mathcal{T}_b, \mathrm{absco}(B_E))$ (Proposition 3.2.9).

We can show that E_+ is closed as follows. Let $\alpha \in \mathbb{R}_{>0}$. Then $E_+ \cap \alpha\mathrm{absco}(B_E) = \alpha\mathrm{co}(\{0\} \cup B_E)$ (Corollary 2.2.9). Now $\alpha\mathrm{co}(\{0\} \cup B_E)$ is the image of $[0, \alpha] \times B_E$ under the continuous map $- \cdot - : \mathbb{R} \times E \to E$, so is compact, and therefore closed. So $E_+ \cap \alpha\mathrm{absco}(B_E)$ is relatively closed in $\mathrm{absco}(B_E)$ for all $\alpha \in \mathbb{R}_{>0}$, which by Lemma 3.2.10 implies it is closed. The map τ is the constant 1 function when restricted to B_E, so τ_{B_E} is continuous. By 3.3.1, $\tau|_{\mathrm{absco} B_E}$ is continuous. Therefore τ is continuous (in $\mathcal{T}_b$) by Proposition 3.2.15, and so $(E, \mathcal{T}_b, E_+, \tau)$ is a Smith base-norm space. □

Recall the category **CCL**, which has pairs (E, X) as objects, where E is a locally convex space and $X \subseteq E$ a compact convex set, and where maps $(E, X) \to (F, Y)$ are simply affine continuous maps $X \to Y$. We have a functor $B : \mathbf{SBNS} \to \mathbf{CCL}$ defined on objects as $B(E) = (E, B_E)$ and on maps as restriction, similar to the definition in and after Proposition 2.2.13 for pre-base-norm spaces and **BConv**.

Proposition 3.3.3. *The functor B : **SBNS** $\to$ **CCL** is an equivalence of categories.*

Proof. We show that B it is faithful, full and essentially surjective. The proof that B is faithful is the same as the proof that $B^{\mathcal{D}} : \mathbf{PreBNS} \to \mathcal{EM}(D)$ is faithful in Proposition 2.4.8.

To show it is full, let $(E, \mathcal{T}, E_+, \tau)$ and $(F, \mathcal{S}, F_+, \sigma)$ be Smith base-norm spaces, and let $g : B_E \to B_F$ be a continuous affine map. it is therefore an affine map and so extends to a trace-preserving (linear) map $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ by Proposition 2.4.8. Since $f|_{B_E} = g$, we know that $f|_{B_E}$ is continuous from $B_E \to (F, \mathcal{S})$, so by Lemma 3.3.1 $f_{\mathrm{absco}B_E}$ is continuous, and since $(E, \mathcal{T}, \mathrm{absco}B_E)$ is a Smith space we apply Proposition 3.2.15 to conclude that f is continuous, and therefore a map in **SBNS**.

To show that it is essentially surjective, let (E, X) be an object of **CCL**, $\mathcal{T}$ being the topology on E, which as all compact sets are bounded (Lemma 0.1.14) is also an element of **BConv**. By Proposition 2.2.13, there exists a pre-base-norm space (F, F_+, τ) with locally convex topology $\mathcal{S}$ and a **BConv** isomorphism $i : (E, X) \to (F, B_F)$ that is relatively continuous from $\mathcal{T}|_X$ to $\mathcal{S}|_{B_F}$. By Proposition 3.3.2, $(F, \mathcal{S}_b, F_+, \tau)$ is a Smith space, whose topology agrees with $\mathcal{S}$ on $\mathrm{absco}B_F$ and therefore on B_F itself. This means the identity mapping $(F, \mathcal{S}, B_F) \to (F, \mathcal{S}_b, B_F)$ is an isomorphism in **CCL**, so composing it with $i : (E, X) \to (F, B_F)$ proves that B is essentially surjective. □

By Theorem 0.4.3, we can find a functor Emb : **CCL** $\to$ **SBNS** (for embedding) such that B and Emb are part of an adjoint equivalence.

3.3.1 Continuous Affine Functions and the Strong Dual

If $(E, X) \in$ **CCL**, we define

$$\mathrm{CAff}(X) = \{a : X \to \mathbb{R} \mid a \text{ affine and continuous}\}.$$

We take $\mathrm{CAff}(X)_+$ to be elements of $\mathrm{CAff}(X)$ with range inside $\mathbb{R}_{\geq 0}$, and its unit to be the constant function with value 1.

Given $f : (E, X) \to (F, Y)$ in **CCL**, we can define

$$\mathrm{CAff}(f)(b) = b \circ f.$$

Proposition 3.3.4. CAff *is a functor from* **CCL** *to* $\mathbf{BOUS}^{\mathrm{op}}$.

Proof. We first show that $\mathrm{CAff}(X)$ is a Banach order-unit space if $(E, X) \in$ **CCL**. We have that $\mathrm{CAff}(X) \subseteq C(X)$, where $C(X)$ is taken as the real-valued continuous functions. That linear combinations of affine functions are affine was already proven in the first part of Proposition 2.4.15, and linear combinations of continuous functions are continuous because addition and multiplication of real numbers are continuous, so $\mathrm{CAff}(X)$ is a linear subspace of $C(X)$. This proves that $\mathrm{CAff}(X)$ is a vector space under the pointwise operations.

We have that $\mathrm{CAff}(X)_+$ is a cone because $C(X, \mathbb{R}_{\geq 0})$ is a cone, the unit element is affine by an argument in Proposition 2.4.15 and continuous because it is constant. The unit element is a strong archimedean unit because it is a strong archimedean unit in $C(X)$, so $\mathrm{CAff}(X)$ is an order-unit space.

To see that $\mathrm{CAff}(X)$ is a Banach space, we show that it is closed in $C(X)$, which is a Banach space. So if $(a_i)_{i \in \mathbb{N}}$ is a sequence of elements of $\mathrm{CAff}(X)$

converging in norm, then we know from the proof in Proposition 2.4.15 that the limit of that sequence is affine. Since $C(X)$ is a Banach space, the limit is also continuous, so the limit of (a_i) is an element of $\mathrm{CAff}(X)$.

We now show that if $f : (E, X) \to (F, Y)$ then $\mathrm{CAff}(f) : \mathrm{CAff}(Y) \to \mathrm{CAff}(X)$ is well-defined and a positive unital map. If we take $b \in \mathrm{CAff}(Y)$, $\mathrm{CAff}(f) = b \circ f$ is an affine map as it is the composite of two affine maps and is continuous because it is the composite of two continuous maps, so is an element of $\mathrm{CAff}(A)$. The linearity of $\mathrm{CAff}(f)$ follows from the pointwiseness of the operations, and the positivity and unitality have the same proof as for Proposition 2.4.16.

Then CAff preserves identity maps because $a \circ \mathrm{id}_X = a$ and preserves composition because $a \circ (g \circ f) = (a \circ g) \circ f$. □

Given a Smith base-norm space $(E, \mathcal{T}, E_+, \tau)$, as $(E, \mathcal{T}, \mathrm{absco}(B_E))$ is a Smith space, the continuous dual is a Banach space E^β (Proposition 3.2.13). We can define E^β_+ to be the dual cone of E_+ (a cone rather than a wedge by Lemma 0.3.8) and the unit $u = \tau$, which is an element of E^β by the definition of Smith base-norm space. Given $\phi \in E^\beta$, we can define $\rho_E(\phi) \in \mathrm{CAff}(B_E)$ to be $\phi|_{B_E}$.

Proposition 3.3.5. *The map $\rho_E : E^\beta \to \mathrm{CAff}(B_E)$ is a linear isomorphism preserving the positive cone and unit both ways. Therefore E^β is a Banach order-unit space for any Smith base-norm space E, with the closed unit ball of E^β being $[-\tau, \tau]$.*

Proof. If $a \in E^\beta$, then $\rho_E(a) \in \mathrm{CAff}(B_E)$, because if a is affine and continuous on E, it is affine and continuous on the subset B_E. The proof in Proposition 2.4.17 shows that $\rho|_E$ is linear and injective, without modification. We also know by the proof in Proposition 2.4.17 that every $a \in \mathrm{BAff}(B_E)$, and therefore every $a \in \mathrm{CAff}(B_E)$ (continuous implies bounded as B_E is compact), extends to a bounded linear map $a' : E \to \mathbb{R}$. Since a' extends a, $a'|_{B_E}$ is continuous in the Smith topologyof E. By Lemma 3.3.1, $a'|_{\mathrm{absco}(B_E)}$ is continuous, and we can then apply Proposition 3.2.15 to deduce that a' is continuous, and therefore an element of $\mathrm{CAff}(B_E)$. Therefore ρ_E is a linear isomorphism.

The proof that it is a positive unital map is the same as in Proposition 2.4.17, as is the proof that the usual dual unit ball is the same as $[-\tau, \tau]$. This shows it is an order-unit space, and by Proposition 3.2.13 it is a Banach order-unit space. □

Theorem 3.3.6. *Restricted to continuous trace-preserving maps, $-^\beta$ is a functor $\mathbf{SBNS} \to \mathbf{BOUS}^{\mathrm{op}}$. We call this functor F^β. Then ρ is a natural isomorphism $F^\beta \Rightarrow \mathrm{CAff} \circ B$.*

Proof. Let $f : (D, \mathcal{T}, D_+, \tau) \to (E, \mathcal{S}, E_+, \sigma)$ be a continuous trace-preserving map. The following diagram commutes

$$\begin{array}{ccc} E^\beta & \xrightarrow{\rho_E} & \mathrm{CAff}(B(E)) \\ {\scriptstyle f^\beta}\downarrow & & \downarrow{\scriptstyle \mathrm{CAff}(B(f))} \\ D^\beta & \xrightarrow[\rho_D]{} & \mathrm{CAff}(B(D)) \end{array}$$

by essentially the same argument given for the diagram commuting in Theorem 2.4.18. In fact, the proof that F^β is therefore a functor and ρ a natural transformation is also essentially the same, and therefore is omitted. $\square$

We can define $G^\sigma : \mathbf{BOUS}^{\mathrm{op}} \to \mathbf{SBNS}$ to be G with the bounded weak-* topology. That is to say, if (A, A_+, u) is a Banach order-unit space, then we know that $G(A, A_+, u)$, which is $(A^*, A^*_+, \mathrm{ev}(u))$, is a base-norm space. Additionally, the unit ball is compact in $\sigma(A^*, A)$ by Banach-Alaoglu, and since $B = A^*_+ \cap \mathrm{ev}(u)^{-1}(1)$ is a $\sigma(A^*, A)$-closed subset of the unit ball, B is compact. Therefore $G^\sigma(A, A_+, u) = (A^*, \sigma(A^*, A)_b, A^*_+, \mathrm{ev}(u))$ is a Smith base-norm space by Proposition 3.3.2. On maps G^σ is defined in the same manner as G, which is to say that if $f : (A, A_+, u) \to (B, B_+, v)$ is a positive unital map and $\psi \in G^\sigma(B)$

$$G^\sigma(f)(\psi) = \psi \circ f$$

Theorem 3.3.7. *G^σ is a functor from $\mathbf{BOUS}^{\mathrm{op}} \to \mathbf{SBNS}$. The restriction of the adjoint equivalence defined by σ and β defines an adjoint equivalence $\mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{SBNS}$.*

Proof. Let $f : (A, A_+, u) \to (B, B_+, v)$ be a positive unital map. Since the definition of G^σ agrees with G, except in topology, we have that $G^\sigma(f)$ is a trace-preserving map of base-norm spaces. By Proposition 1.2.8, f is a map in $\mathbf{Ban}_1$, and as the definition of G^σ agrees with -^σ, we have that $G^\sigma(f)$ is a continuous map of Smith spaces by Proposition 3.2.21. The proof that G^σ preserves identity maps and composition then follows from the proof that G and -^σ are functors.

Following Theorem 3.2.22 and Proposition 2.5.3, we define

$$\begin{aligned} \eta_E &: E \to G^\sigma(F^\beta(E)) & \qquad \epsilon_A &: A \to F^\beta(G^\sigma(A)) \\ \eta_E(x)(a) &= a(x) & \epsilon_A(a)(\phi) &= \phi(a). \end{aligned}$$

The underlying space of $G^\sigma(F^\beta(E))$ is $E^{\beta\sigma}$ and the underlying space of $F^\beta(G^\sigma(A))$ is $A^{\sigma\beta}$. By Theorem 3.2.22 these maps are linear homeomorphisms of the underyling topological vector spaces.

We will show that η_E is an isomorphism of Smith base-norm spaces and ϵ_A an isomorphism of Banach order-unit spaces.

The proof that η_E is positive is similar to the proof in Proposition 2.5.3, in short, if $x \in E_+$ and $a \in E^\beta_+$, we have $\eta_E(x)(a) = a(x) \geq 0$, so $\eta_E(x) \in E^{\beta\sigma}_+$. If, on the other hand, we start with $\Phi \in E^{\beta\sigma}_+$, we know from the bijectivity of η_E that there exists an $x \in E$ such that $\eta_E(x) = \Phi$. The positivity of Φ implies that for all $a \in E_+^\beta$, $\eta_E(x)(a) \geq 0$. Expanding the definition, we have that $a(x) \geq 0$ for all $a \in E_+^\beta$. As E_+ is a closed cone and $E^\beta = E^*$, we can use Lemma 0.3.15 to deduce that $x \in E_+$. We have shown that $\eta_E(E_+) = E_+^{\beta\sigma}$, and therefore the inverse of η_E is also positive.

The proof that η_E is trace-preserving is similar to that in Proposition 2.5.3 but as it is short we can show it here. The trace of $G^\sigma(F^\beta(E))$ is $\mathrm{ev}(\tau)$. If $x \in E$ we have

$$(\mathrm{ev}(\tau) \circ \eta_E)(x) = \eta_E(x)(\tau) = \tau(x),$$

so $\mathrm{ev}(\tau) \circ \eta_E = \tau$. It follows that $\tau \circ \eta_E^{-1} = \mathrm{ev}(\tau)$, so the inverse is also trace-preserving.

The proof that ϵ_A and its inverse are positive is similar to the proof for η_E, except using the fact that A_+ is norm-closed by Lemma A.5.3. The proof of unitality is as follows. Let $\phi \in A^\sigma$:

$$\epsilon_A(u)(\phi) = \phi(u) = \mathrm{ev}(u)(\phi),$$

so $\epsilon_A(u) = \mathrm{ev}(u)$, the unit of $F^\beta(G^\sigma(A))$. If a map is unital, then its inverse must also be, so ϵ_A^{-1} is also unital.

In each case we know by Theorem 3.2.22 that the naturality diagrams commute with maps that are only linear and continuous, so they commute *a fortiori* for maps of Smith base-norm spaces and maps of Banach order-unit spaces. The commutativity of the diagrams to show that this is an adjoint equivalence also follows in this way. □

We define $\mathrm{Stat} : \mathbf{BOUS}^{\mathrm{op}} \to \mathbf{CCL}$ to be $B \circ G^\sigma$. This is an equivalence of categories. In fact:

Theorem 3.3.8. *The functor* CAff *is a left adjoint to* Stat*. Therefore this adjunction is an adjoint equivalence.*

Proof. We use Theorem 0.4.1 (v) to define the adjunction by defining a unit and a counit. The counit should be a natural transformation $\epsilon : \mathrm{CAff} \circ \mathrm{Stat} \Rightarrow \mathrm{Id}_{\mathbf{BOUS}}$ in $\mathbf{BOUS}^{\mathrm{op}}$, *i.e.* $\epsilon : \mathrm{Id}_{\mathbf{BOUS}} \Rightarrow \mathrm{CAff} \circ \mathrm{Stat}$ in $\mathbf{BOUS}$. If we temporarily use ϵ' to refer to the counit in Theorem 3.3.7, and use the natural transformation ρ from Theorem 3.3.6, we can define

$$\epsilon = \rho G^\sigma \circ \epsilon' : \mathrm{Id}_{\mathbf{BOUS}} \Rightarrow \mathrm{CAff} \circ \mathrm{Stat}.$$

By defining it in this way, it is already proven that ϵ is well defined and natural. We can also expand the definition for $A \in \mathbf{BOUS}$, $a \in A$ and $\phi \in \mathrm{Stat}(A)$:

$$\epsilon_A(a)(\phi) = \rho_{G^\sigma(A)}(\epsilon'_A(a))(\phi) = \epsilon'_A(a)(\phi) = \phi(a),$$

because ρ is just restriction to the base of a Smith base-norm space.

We define the unit as follows, for $(E, X) \in \mathbf{CCL}$, $x \in X$, and $a \in \mathrm{CAff}(X)$:

$$\eta_X(x)(a) = a(x).$$

We prove that this is defined correctly as follows.

- $\eta_X(x)$ is a state on $\mathrm{CAff}(X)$:

  We see that $\eta_X(x)$ preserves addition and scalar multiplication by the pointwiseness of the definition of those operations on $\mathrm{CAff}(X)$. Similarly, since the positive cone of $\mathrm{CAff}(X)$ is defined to be exactly those continuous affine functions taking nonnegative values at every point of X, we have that $\eta_X(x)$ is positive. For unitality, we have $\eta_X(x)(1) = 1(x) = 1$.

- η_X is an affine map $X \to \mathrm{Stat}(\mathrm{CAff}(X))$:

  Let $x, y \in X$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}
\eta_X(\alpha x + (1-\alpha)y) &= a(\alpha x + (1-\alpha)y) \\
&= \alpha a(x) + (1-\alpha)a(y) && a \text{ affine} \\
&= \alpha\eta_X(x)(a) + (1-\alpha)\eta_X(y)(a) \\
&= (\alpha\eta_X(x) - (1-\alpha)\eta_X(y))(a)
\end{aligned}$$

- η_X is continuous from X to the weak-* topology on $\mathrm{Stat}(\mathrm{CAff}(X))$:

  We use preservation of convergence of nets as the definition of continuity. Let $(x_i)_{i\in I}$ be a net converging to x in X. For each $a \in \mathrm{CAff}(X)$ we have

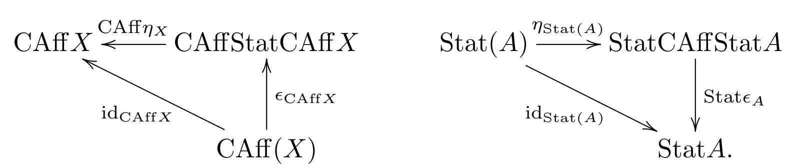
$$\eta_X(x_i)(a) = a(x_i) \to a(x) = \eta_X(x)(a)$$

  because a is continuous. As this is true for all $a \in \mathrm{CAff}(X)$, we have that $\eta_X(x_i) \to \eta_X(x)$ in the weak-* topology, and therefore η_X is continuous.

We can show that η is natural as follows. The commutativity of the naturality diagram for a map $f : X \to Y$ in **CCL** is equivalent to $\eta_Y \circ f = \mathrm{Stat}(\mathrm{CAff}(f)) \circ \eta_X$. If we let $x \in X$ and $b \in \mathrm{CAff}(Y)$, we have

$$\begin{aligned}
(\mathrm{Stat}(\mathrm{CAff}(f)) \circ \eta_X)(x)(b) &= \mathrm{Stat}(\mathrm{CAff}(f))(\eta_X(x))(b) = \eta_X(x)(\mathrm{CAff}(f)(b)) \\
&= \mathrm{CAff}(f)(b)(x) = b(f(x)) = \eta_Y(f(x))(b) \\
&= (\eta_Y \circ f)(x)(b)
\end{aligned}$$

The unit and counit diagrams are



To show that the left hand diagram commutes, let $a \in \mathrm{CAff}(X)$ and $x \in X$ in the following:

$$\mathrm{CAff}(\eta_X)(\epsilon_{\mathrm{CAff}(X)}(a))(x) = \epsilon_{\mathrm{CAff}(X)}(a)(\eta_X(x)) = \eta_X(x)(a) = a(x),$$

so $(\mathrm{CAff}(\eta_X) \circ \epsilon_{\mathrm{CAff}(X)})(a) = a$ for all $a \in \mathrm{CAff}(X)$, and therefore the diagram commutes.

For the right hand diagram, let $\phi \in \mathrm{Stat}A$ and $a \in A$ in the following:

$$\mathrm{Stat}(\epsilon_A)(\eta_{\mathrm{Stat}(A)}(\phi))(a) = \eta_{\mathrm{Stat}(A)}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) = \phi(a).$$

We therefore have that $\mathrm{CAff} \dashv \mathrm{Stat}$. We already have that $F^\beta \circ \mathrm{Emb} \dashv B \circ G^\sigma$ by composing the adjoint equivalences arising from Proposition 3.3.3 and Theorem 3.3.7, and so $(\mathrm{CAff}, \mathrm{Stat}, \eta, \epsilon)$ form an adjoint equivalence by Lemma 0.4.4. □

We call the above adjoint equivalence *Kadison duality*, because it was Kadison who first proved that ϵ_A was an isomorphism [70, Lemma 2.5][71, Lemma 4.3, Remark 4.4]. A more modern proof that ϵ_A is an isomorphism can also be found in [4, Theorem II.1.8]. The name Kadison duality was first used publicly in [59].

We now have three sides of the square (3.1). We define $\mathrm{Stat} : \mathbf{BEMod}^{\mathrm{op}} \to \mathbf{CCL}$, using the same name as $\mathrm{Stat} : \mathbf{BOUS}^{\mathrm{op}} \to \mathbf{CCL}$, to be $B(G^\sigma(A))$ for all Banach effect modules of the form $[0,1]_A$. As $[0,1]_{\text{-}}$ is an equivalence, this can be extended to all of $\mathbf{BEMod}^{\mathrm{op}}$, and all such extensions are naturally isomorphic. This also ensures that $\mathrm{Stat} \circ [0,1]_{\text{-}} = B \circ G^\sigma$.

We define $\mathrm{CAff}(X, [0,1])$ for X an object of $\mathbf{CCL}$ to be

$$\mathrm{CAff}(X, [0,1]) = \{a \in \mathrm{CAff}(X) \mid \forall x \in X. 0 \le a(x) \le 1\}.$$

Since the order on $\mathrm{CAff}(X)$ is pointwise and 0 and u are given by constant functions, it is clear that $\mathrm{CAff}(X, [0,1]) = [0,1]_{\mathrm{CAff}(X)}$.

On maps $f : (E, X) \to (F, Y)$ in $\mathbf{CCL}$ we define

$$\mathrm{CAff}(f, [0,1])(b) = b \circ f.$$

This definition agrees with $\mathrm{CAff}(f)$, so we have that $\mathrm{CAff}(\text{-}, [0,1]) = [0,1]_{\text{-}} \circ \mathrm{CAff}$, and is therefore a functor.

Theorem 3.3.9. *In* (3.1) *we have* $\mathrm{CAff}(\text{-}, [0,1]) \circ B \cong [0,1]_{\text{-}} \circ F^\beta$ *and that* $\mathrm{CAff}(\text{-}, [0,1])$ *and* Stat *define an equivalence between* $\mathbf{CCL}$ *and* $\mathbf{BEMod}^{\mathrm{op}}$*.*

Proof. We have a natural isomorphism $\rho : F^\beta \Rightarrow \mathrm{CAff} \circ B$ from Theorem 3.3.6. Therefore $[0,1]_\rho : [0,1]\text{-} \circ F^\beta \Rightarrow [0,1]_{\text{-}} \circ \mathrm{CAff} \circ B = \mathrm{CAff}(\text{-}, [0,1]) \circ B$, which is the isomorphism we need.

To prove that $\mathrm{CAff}(\text{-}, [0,1])$ and Stat define an equivalence, we show that $\mathrm{CAff}(\text{-}, [0,1]) \circ \mathrm{Stat} \cong \mathrm{Id}_{\mathbf{BEMod}}$ and $\mathrm{Stat} \circ \mathrm{CAff}(\text{-}, [0,1]) \cong \mathrm{Id}_{\mathbf{CCL}}$. We reason as follows

$$\begin{aligned} \mathrm{CAff}(\text{-}, [0,1]) \circ B &\cong [0,1]_{\text{-}} \circ F^\beta && \Leftrightarrow \\ \mathrm{CAff}(\text{-}, [0,1]) \circ B \circ \mathrm{Emb} &\cong [0,1]_{\text{-}} \circ F^\beta \circ \mathrm{Emb} && \Leftrightarrow \\ \mathrm{CAff}(\text{-}, [0,1]) &\cong [0,1]_{\text{-}} \circ F^\beta \circ \mathrm{Emb}, \end{aligned}$$

as Emb is an inverse for B. Similarly, we have

$$\begin{aligned} \mathrm{Stat} \circ [0,1]_{\text{-}} &= B \circ G^{\sigma} && \Leftrightarrow \\ \mathrm{Stat} \circ [0,1]_{\text{-}} \circ \mathcal{T} &= B \circ G^{\sigma} \circ \mathcal{T} && \Leftrightarrow \\ \mathrm{Stat} &\cong B \circ G^{\sigma} \circ \mathcal{T}, \end{aligned}$$

as $\mathcal{T}$ is an inverse for $[0,1]_{\text{-}}$.

We then have

$$\begin{aligned} \mathrm{Stat} \circ \mathrm{CAff}(\text{-}, [0,1]) &\cong B \circ G^{\sigma} \circ \mathcal{T} \circ [0,1]_{\text{-}} \circ F^{\beta} \circ \mathrm{Emb} \\ &\cong B \circ G^{\sigma} \circ F^{\beta} \circ \mathrm{Emb} && \text{Theorem 1.2.9} \\ &\cong B \circ \mathrm{Emb} && \text{Theorem 3.3.7} \\ &\cong \mathrm{Id}_{\mathbf{CCL}} && \text{Proposition 3.3.3.} \end{aligned}$$

On the other side, we have

$$\begin{aligned} \mathrm{CAff}(\text{-}, [0,1]) \circ \mathrm{Stat} &\cong [0,1]_{\text{-}} \circ F^{\beta} \circ \mathrm{Emb} \circ B \circ G^{\sigma} \circ \mathcal{T} \\ &\cong [0,1]_{\text{-}} \circ F^{\beta} \circ G^{\sigma} \circ \mathcal{T} && \text{Proposition 3.3.3} \\ &\cong [0,1]_{\text{-}} \circ \mathcal{T} && \text{Theorem 3.3.7} \\ &\cong \mathrm{Id}_{\mathbf{BEMod}} && \text{Theorem 1.2.9} \end{aligned}$$

□

3.4 Compact Effect Modules and Smith Order-Unit Spaces

In this section, we reverse which kind of space has a Smith topology with respect to the previous section. We begin with the definition of a Smith order-unit space. A *Smith order-unit space* is a quadruple $(E, \mathcal{T}, E_+, u)$ where $(E, \mathcal{T})$ is a locally convex topology, E_+ a closed positive cone, (E, E_+, u) an order-unit space, such that $[-u, u]$ is compact and $\mathcal{T}$ is a Smith space topology with respect to the compact barrel $[-u, u]$. Unital and subunital maps of Smith order-unit spaces are simply unital and subunital maps of the underlying order-unit spaces that are continuous, and these maps define the categories **SOUS** and $\mathbf{SOUS}_{\leq 1}$.

Proposition 3.4.1. *If (A, A_+, u) is a partially ordered vector space with strong order unit, $\mathcal{T}$ a locally convex topology on A in which $[0, u]$ is compact, then $(A, \mathcal{T}_b, A_+, \tau)$ is a Smith order-unit space, where $\mathcal{T}_b$ is taken with respect to the compact barrel $[-u, u]$.*

Proof. We first show that $[-u, u]$ is compact by observing that $2[0, u] - u = [0, u]$ (Lemma 0.2.2) and observing that scalar multiplication and addition are continuous and therefore map compact sets to compact sets.

Since $[-u, u]$ is affinely isomorphic to $[0, u]$, it is convex, and it is balanced because $-[-u, u] = [-u, u]$ so is absolutely convex by Lemma A.3.1. It is absorbent by the definition of a strong order unit, so $[-u, u]$ is a compact barrel.

We can therefore give A the Smith topology $\mathcal{T}_b$ with respect to $[-u, u]$ (Proposition 3.2.9).

We can now show that A_+ is closed in this topology as follows. We have that $A_+ \cap [-\alpha u, \alpha u] = [0, \alpha u]$ for all $\alpha \in \mathbb{R}_{>0}$, and by Lemma 0.2.2, $A_+ \cap \alpha[-u, u] = \alpha[0, u]$. Since multiplication by a scalar is continuous, $\alpha[0, u]$ is compact, and therefore closed. Therefore A_+ is closed, by Lemma 3.2.10 (ii). By Lemma 3.2.3 A_+ is also norm-closed, and this implies (A, A_+, u) is archimedean by Lemma A.5.3. This proves it is an order-unit space, and the facts already proven show it is a Smith order-unit space. □

We can now define compact effect modules. As with compact convex sets, we deal first with the "concrete" definition, and later give an alternative definition via monads. We know that for every effect module A, there is a partially ordered vector space with strong unit (E, E_+, u) and an **EMod** isomorphism $A \cong [0, u]_E$. We know by Lemma A.4.1 that effect modules have an intrinsic notion of convex combination, which maps to convex combinations in $[0, u]_E$. We define the category **CEMod** to have objects (E, A), where E is a locally convex space and A an effect module structure on a compact convex subset of E such that for all $x, y \in A$ and $\alpha \in [0, 1]$, we have that $\alpha x \ovee (1 - \alpha)y$, calculated using the effect module structure, equals $\alpha x + (1 - \alpha)y$, calulated using the vector space structure of E. The maps in **CEMod** are effect module maps that are also continuous, and by Lemma A.4.1 they are affine.

We first prove a lemma about elements of **CEMod**.

Lemma 3.4.2. *Let $(E, A) \in$ **CEMod**. If $a, b \in A$ such that $a \perp b$, we have*

$$a \ovee b = a + b,$$

where $+$ is the vector space addition for E. We also have that $a^\perp = 1 - a$, 1 being the unit element of A. If $0_E = 0_A$, then if $\alpha \in [0, 1]$, we have $\alpha \cdot_A a = \alpha \cdot_E a$, where the subscript on the $\cdot$ and 0 indicates which structure it refers to.

Proof. For the additive part, we reason as follows, with all scalar multiplications being in A, not E.

$$\begin{aligned}
a \varovee b &= \frac{1}{2}(a \varovee b) \varovee \frac{1}{2}(a \varovee b) \\
&= \frac{1}{2}(a \varovee b) + \frac{1}{2}(a \varovee b) && \text{convex combinations} \\
&= \left(\frac{1}{2}a \varovee \frac{1}{2}b\right) + \left(\frac{1}{2}a \varovee \frac{1}{2}b\right) && \text{effect module axiom} \\
&= \left(\frac{1}{2}a + \frac{1}{2}b\right) + \left(\frac{1}{2}a + \frac{1}{2}b\right) && \text{convex combinations} \\
&= \left(\frac{1}{2}a + \frac{1}{2}a\right) + \left(\frac{1}{2}b + \frac{1}{2}b\right) \\
&= \left(\frac{1}{2}a \varovee \frac{1}{2}a\right) + \left(\frac{1}{2}b \varovee \frac{1}{2}b\right) && \text{convex combinations} \\
&= a + b,
\end{aligned}$$

the last step using the effect module axioms.

We now have that because $a \varovee a^\perp = 1$, $a + a^\perp = 1$, and so $a^\perp = 1 - a$.

We now assume that the zero of A is the same element as the zero of E, or in our notation, $0_A = 0_E$. Then

$$\begin{aligned}
\alpha \cdot_A a &= \alpha \cdot_A a \varovee 0_A && \text{effect algebra axiom} \\
&= \alpha \cdot_A a \varovee (1-\alpha) \cdot_A 0_A && \text{Lemma A.4.2} \\
&= \alpha \cdot_E a + (1-\alpha) \cdot_E 0_A && \text{convex combinations} \\
&= \alpha \cdot_E a + (1-\alpha) \cdot_E 0_E && 0_A = 0_E \\
&= \alpha \cdot_E a.
\end{aligned}$$

□

For each order-unit space (A, A_+, u), we have seen that we have an effect module $[0,1]_A$, and this in fact defines a functor **OUS** $\to$ **EMod**. We can now deal with the analogue of this for Smith order-unit spaces and compact effect modules.

We can see that if $(A, \mathcal{T}, A_+, u)$ is a Smith order-unit space, then $(A, [0,1]_A)$ is an object of **CEMod**. Additionally, if $f : A \to B$ is a unital morphism of Smith spaces, we already know $f|_{[0,1]_A}$ is a morphism of effect modules, and it is a **CEMod** map because it is continuous. As this is simply restriction of functions, $[0,1]_{-}$ is a functor **SOUS** $\to$ **CEMod**.

Theorem 3.4.3. *The functor* $[0,1]_{-}$: **SOUS** $\to$ **CEMod** *is an equivalence.*

Proof. In the following, let $(A, \mathcal{T}, A_+, u)$ and $(B, \mathcal{S}, B_+, v)$ be Smith order-unit spaces.

The functor $[0,1]_{-}$ is faithful because for any order-unit space, $[0,1]_A$ spans A, so if $f|_{[0,1]_A} = g|_{[0,1]_A}$, then $f = g$ by linearity.

Now let $g : [0,1]_A \to [0,1]_B$ be a continuous effect module homomorphism. We know by the fullness of $[0,1]_{\text{-}} : \mathbf{poVectu} \to \mathbf{EMod}$ that there is an $f : A \to B$ that is linear, positive and unital such that $[0,1]_f = g$. Therefore $f|_{[0,1]_A}$ is continuous. However, we need to show that f is continuous. We first show that $f|_{[0,2]_A}$ is continuous.

Let $V \subseteq B$ be an open set. Since B is a topological vector space $\frac{1}{2}B$ is also open. By continuity of g, we have that $f^{-1}(\frac{1}{2}V) \cap [0,u] = U' \cap [0,u]$ for some U' an open subset of A. Multiplying both sides by 2, we get

$$2f^{-1}\left(\frac{1}{2}V\right) \cap [0,2u] = 2U' \cap [0,2u],$$

and using the linearity of f this shows

$$f^{-1}(V) \cap [0,2u] = 2U' \cap [0,2u].$$

Since $2U'$ is an open subset of A, this implies that $f|_{[0,2]_A}$ is continuous. From here, we can show that $f|_{[-1,1]_A}$ is continuous.

So again, let $V \subseteq B$ be an open set. We see that $V + f(u)$ is also open, as B is a topological vector space. We therefore have that $f^{-1}(V + f(u)) \cap [0,2u] = U' \cap [0,2u]$ for some open $U' \subseteq A$. We can then subtract u from both sides and get

$$\begin{aligned}(f^{-1}(V + f(u)) \cap [0,2u]) - u = (U' \cap [0,2u]) - u \qquad &\Leftrightarrow\\ (f^{-1}(V) + u - u) \cap [-u,u] = (U' - u) \cap [-u,u] \qquad &\Leftrightarrow\\ f^{-1}(V) \cap [-u,u] = (U' - u) \cap [-u,u]. &\end{aligned}$$

We have that $U' - u$ is an open subset of A, so $f|_{[-1,1]_A}$ is continuous. We then apply Proposition 3.2.15 to conclude that f is continuous, and therefore a map in **SOUS**, proving the fullness.

We now move on to the final stage, proving that $[0,1]_{\text{-}}$ is essentially surjective. Let $(E,A) \in \mathbf{CEMod}$. We need to find $(A', \mathcal{T}, A'_+, u) \in \mathbf{SOUS}$ and a continuous effect module isomorphism $i : A \to [0,1]_{A'}$. For purposes of disambiguation, we will at first use 0_A to refer to the 0 element of A and 0_E to that of E. We can then redefine A to be $A - 0_A$ and make $\text{-} - 0_A$ an isomorphism $(E,A) \cong (E, A - 0_A)$, so that $0_A = 0_E$, which we now refer to as 0 again.

Define $A' = \text{span}(A)$, $A'_+ = \bigcup_{\alpha \in \mathbb{R}_{>0}} \alpha A$ and $u = 1_A$. We can show that $[0,1]_{A'} = A$ right away:

$$[0,1]_{A'} = \{a \in \text{span}(A) \mid a \in A'_+ \text{ and } u - a \in A'_+\} = \{a \in A'_+ \mid u - a \in A'_+\},$$

as $A'_+ \subseteq \text{span}(A)$ by definition. Suppose that a is an element of this set, *i.e.* that $a = \lambda a'$ where $a' \in A$ and $\lambda \in \mathbb{R}_{>0}$, and $u - a = \mu a''$ where $a'' \in A$ and $\mu \in \mathbb{R}_{>0}$, which is to say that $a = u - \mu a''$. Eliminating a, we get that $\lambda a' = u - \mu a''$, or $\lambda a' + \mu a'' = u$. We know that $\lambda + \mu > 0$ and we can take $n = \lceil \lambda + \mu \rceil \geq 1$. Then $\frac{\lambda}{n} + \frac{\mu}{n} \leq 1$, so the equation

$$\frac{\lambda}{n}a' \ovee \frac{\mu}{n}a'' = \frac{1}{n}u$$

holds in A (Lemma A.4.1). We can then add this equation to itself n times and rearrange the terms using commutativity and associativity to get

$$\left(\frac{\lambda}{n}a' \ovee \cdots \ovee \frac{\lambda}{n}a'\right) \ovee \left(\frac{\mu}{n}a'' \ovee \frac{\mu}{n}a''\right) = u,$$

and we therefore have that these repeated additions are defined as elements of A. We can apply Lemma 3.4.2 to conclude that

$$\frac{\lambda}{n}a' \ovee \cdots \ovee \frac{\lambda}{n}a' = \lambda a'$$

by replacing the effect module operations by the vector space ones, and we therefore have $a = \lambda a' \in A$. This shows that $[0,1]_A \subseteq A$. Now if $a \in A$, we have that $a \in A_+$, with $\lambda = 1$, and $a \in u - A_+$ because $a = 1 - a^{\perp}$ by Lemma 3.4.2, so $A \subseteq [0,1]_A$. We now show that (A', A'_+, u) is a partially ordered vector space with strong order unit.

- A'_+ is a cone:

  For closure under addition, let $a, b \in A'_+$, *i.e.* $a = \lambda a'$ and $b = \mu b'$ for $a', b' \in A$, $\lambda, \mu \in \mathbb{R}_{>0}$. By Lemma A.4.1, we have

  $$\frac{\lambda}{\lambda+\mu}a' \ovee \frac{\mu}{\lambda+\mu}b' \in A,$$

  and

  $$\begin{aligned} a + b = \lambda a' + \mu b' &= (\lambda+\mu)\left(\frac{\lambda}{\lambda+\mu}a' + \frac{\mu}{\lambda+\mu}b'\right) \\ &= (\lambda+\mu)\left(\frac{\lambda}{\lambda+\mu}a' \ovee \frac{\mu}{\lambda+\mu}b'\right), \end{aligned}$$

  by preservation of convex combinations. We have therefore shown $a + b \in A'_+$.

  We have that $0 \in A \subseteq A'_+$ directly. To show that A'_+ is closed under positive scalar multiplication, we separate into two cases. Let $a = \lambda a'$ with $a \in A$ as before. If $\mu \in \mathbb{R}_{>0}$, then $\mu a = \mu\lambda a'$, and since $\mu\lambda > 0$ we have that $\mu a \in A'_+$. The other case is when $\mu = 0$. In this case, $\mu a = 0$, which is in A'_+ as we already showed.

- u is a strong order unit:

  We show this in two steps using Lemma A.5.1. We first show that $A'_+ - A'_+ = A'$, as follows. Since A' is the span of A, we have that every $a \in A'$ can be expressed as $\sum_{i \in I} \alpha_i a_i$, where $\alpha_i \in \mathbb{R} \setminus 0$[4], $a_i \in A$, and I is a finite set. We then define

  $$I_+ = \{i \in I \mid \alpha_i > 0\} \qquad\qquad I_- = \{i \in I \mid \alpha_i < 0\},$$

[4] We can exclude any zero terms without affecting the value of the sum.

and we have $I_+ \cup I_- = I$. Therefore

$$a = \sum_{i \in I_+} \alpha_i a_i - \sum_{i \in I_-} (-\alpha_i) a_i,$$

and this expresses a as a difference of two elements of A'_+.

We can then see that for $\alpha \in \mathbb{R}_{>0}$, $\alpha[0,u] = [0, \alpha u]$ as follows:

$$\begin{aligned} x \in \alpha[0,u] &\Leftrightarrow \alpha^{-1}x \in [0,u] \\ &\Leftrightarrow \alpha^{-1}x \in A'_+ \text{ and } \alpha^{-1}x \in u - A'_+ \\ &\Leftrightarrow \alpha^{-1}x \in A'_+ \text{ and } u - \alpha^{-1}x \in A'_+ \\ &\Leftrightarrow x \in A'_+ \text{ and } \alpha u - x \in A'_+ && A'_+ \text{ a cone} \\ &\Leftrightarrow x \in [0, \alpha u]. \end{aligned}$$

Therefore

$$\bigcup_{n \in \mathbb{N}} [0, nu] = \bigcup_{\alpha \in \mathbb{R}_{>0}} [0, \alpha u] = \bigcup_{\alpha \in \mathbb{R}_{>0}} \alpha[0,u] = A'_+,$$

so u is a strong order unit.

We then use the compactness of $A = [0,u]$ to apply Proposition 3.4.1, defining $\mathcal{T}$ to be the Smithization of the original (subspace) topology on A, to get that $(A', \mathcal{T}, A'_+, u)$ is a Smith order-unit space. □

Now we define G^β. On a Smith order-unit space $(A, \mathcal{T}, A_+, u)$ we define

$$G^\beta(A) = (A^\beta, A^\beta_+, \mathrm{ev}(u)),$$

and for a continuous unital or subunital map $f : (A, \mathcal{T}, A_+, u) \to (B, \mathcal{S}, B_+, v)$ and $\phi \in B^\beta$

$$G^\beta(f)(\phi) = \phi \circ f$$

Proposition 3.4.4. *G^β is a functor* $\mathbf{SOUS}^{\mathrm{op}}_{\leq 1} \to \mathbf{BBNS}_{\leq 1}$ *and* $\mathbf{SOUS}^{\mathrm{op}} \to \mathbf{BBNS}$.

Proof. We first show that $G^\beta(A, \mathcal{T}, A_+, u) = (A^\beta, A^\beta_+, \mathrm{ev}(u))$ is a Banach base-norm space. In aid of this, we define $F = A^\beta_+ - A^\beta_+ \subseteq A^\beta$, $B^{\leq 1} = A^\beta_+ \cap \mathrm{ev}(u)^{-1}((-\infty, 1]) = \{\phi \in A^\beta \mid \phi(u) \leq 1 \text{ and } \forall a \in A_+.\phi(a) \geq 0\}$, and $V = \mathrm{co}(B^{\leq 1} \cup -B^{\leq 1})$.

We already know that A^β is a Banach space with unit ball $[-u,u]^{\mathrm{o}}$ by Proposition 3.2.13. We show that $B^{\leq 1} = A^\beta_+ \cap [-u,u]^{\mathrm{o}}$ as follows. If $\phi \in [-u,u]^{\mathrm{o}}$, we can conclude from $u \in [-u,u]$ that $\mathrm{ev}(u)(\phi) = \phi(u) \leq 1$, so $\phi \in \mathrm{ev}(u)^{-1}((-\infty, 1])$. Therefore $A^\beta_+ \cap [-u,u]^{\mathrm{o}} \subseteq B^{\leq 1}$. For the other direction, observe that if $\phi \in A^\beta_+$, then it preserves positive elements as a map $\phi : A \to \mathbb{R}$, so is a monotone map. Therefore if $\phi \in A^\beta_+ \cap \mathrm{ev}(u)^{-1}((-\infty, 1])$, and $a \in [-u,u]$, we have $\phi(a) \leq \phi(u) = \mathrm{ev}(u)(\phi) \leq 1$, so $\phi \in [-u,u]^{\mathrm{o}}$.

As A^β_+ and $[-u,u]^{\mathrm{o}}$ are closed in A^β, $B^{\leq 1}$ is closed, and therefore complete, and so is σ-convex (by Lemma 0.1.19). We can therefore apply Lemma 2.2.15 to deduce that $(F, \|\text{-}\|_V)$ is a Banach space. Consider the inclusion map $i : F \to A^\beta$, which is a contraction as $i(V) \subseteq [-u,u]^{\mathrm{o}}$ because $B^{\leq 1} \subseteq [-u,u]^{\mathrm{o}}$ and $[-u,u]^{\mathrm{o}}$ is absolutely convex. The map $i^\sigma : A^{\beta\sigma} \to F^\sigma$ exists as a map of Smith spaces. We aim to show that this is a $\mathbf{Smith}_1$ isomorphism, from which we could conclude that $F = A^\beta$.

First consider V as a subset of A^β, in its pairing with A. We can show that $V^{\mathrm{o}} = [-u,u]$ as follows.

$$\begin{aligned} V^{\mathrm{o}} &= \mathrm{co}(B^{\leq 1} \cup -B^{\leq 1})^{\mathrm{o}} \\ &= \mathrm{absco}(B^{\leq 1})^{\mathrm{o}} \\ &= {B^{\leq 1}}^{|\mathrm{o}|} && \text{Lemma 0.3.11 (iv)} \\ &= \{a \in A \mid \forall \phi \in B^{\leq 1}. -1 \leq \phi(a) \leq 1\} = X \end{aligned}$$

We now show that this set X that we have just defined is equal to

$$Y = \{a \in A. \forall \phi \in A^\beta_+. -\phi(u) \leq a \leq \phi(u)\}.$$

- $X \subseteq Y$:

  Let $a \in X$ and suppose that $\phi \in A^\beta_+$. Then if $\phi(u) = 0$, by Lemma A.5.4 $\phi = 0$, so $-\phi(u) \leq \phi(a) \leq \phi(u)$ because they are all zero. If $\phi(u) \neq 0$, then $\phi(u) > 0$. Let $\alpha = \phi(u)$. The map $\alpha^{-1}\phi \in B^{\leq 1}$, and therefore

  $$-1 \leq \alpha^{-1}\phi(a) \leq 1.$$

  Multiplying through by α and substituting it for its definition we get

  $$-\phi(u) \leq \phi(a) \leq \phi(u)$$

  as required.

- $Y \subseteq X$: If $a \in Y$, and $\phi \in B^{\leq 1}$, then $\phi(u) \leq 1$ by definition. Therefore

  $$-1 \leq -\phi(u) \leq \phi(a) \leq \phi(u) \leq 1.$$

So far we have shown that $V^{\mathrm{o}} = Y$. Now

$$\begin{aligned} Y &= \{a \in A. \forall \phi \in A^\beta_+. -\phi(u) \leq \phi(a) \leq \phi(u)\} \\ &= \{a \in A. (\forall \phi \in A^\beta_+. \phi(u+a) \geq 0) \text{ and } (\forall \phi \in A^\beta_+. \phi(u-a) \geq 0\} \\ &= \{a \in A. u + a \in A_+ \text{ and } u - a \in A_+\} && \text{Lemma 0.3.15} \\ &= [-u,u]. \end{aligned}$$

Therefore $V^{\mathrm{oo}} = [-u,u]^{\mathrm{o}} = U$. Therefore U is the $\sigma(A^\beta, A)$-closure of V (Corollary 0.3.10), and therefore the $\sigma(A^\beta, A^{\beta\sigma})$-closure of V (Proposition 3.2.17), which by Proposition 0.3.4 is in fact the $\|\text{-}\|_U$-closure. Therefore if

$\phi \in A^\beta$, there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha^{-1}\phi \in U$. There is therefore a sequence $(\psi_i)_{i\in\mathbb{N}}$ in V, and therefore in F, converging in $\|\text{-}\|_U$ to $\alpha^{-1}\psi$. Therefore $\alpha\psi_i \to \phi$, and so F is $\|\text{-}\|_U$-dense in A^β. As the inclusion map i is also injective, by Proposition 3.2.25 i^σ is injective with dense image.

If we consider the pairings between F and F^σ and A^β and $A^{\beta\sigma}$, the map $i : F \to A^\beta$ has adjoint $i^\sigma : A^{\beta\sigma} \to F^\sigma$. Considering the polars with respect to these dualities, we have

$$(i^\sigma)^{-1}(V^\text{o}) = i(V)^\text{o} = U^\text{o}$$

by Lemma 0.3.14 and the fact that $U = \text{cl}\,(V)$. Therefore

$$i^\sigma(U^\text{o}) = i^\sigma((i^\sigma)^{-1}(V^\text{o})) = i^\sigma(A^{\beta\sigma}) \cap V^\text{o}. \tag{3.10}$$

As U^o is compact, being the unit ball of $A^{\beta\sigma}$, $i^\sigma(U^\text{o})$ is compact, and therefore closed. We have therefore shown that $i^\sigma(A^{\beta\sigma}) \cap V^\text{o}$ is closed. By multiplying by α and using the fact that $i^\sigma(A^{\beta\sigma})$ is a subspace, we see that $i^\sigma(A^{\beta\sigma}) \cap \alpha V^\text{o}$ is closed. By Lemma 3.2.10, $i^\sigma(A^{\beta\sigma})$ is closed, and as we already showed it was dense in F^σ, we have that i^σ is surjective. Therefore i^σ is a continuous linear bijection of Smith spaces, so therefore is an isomorphism of Smith spaces by Corollary 3.2.19.

All together, we have seen that i^σ is an isomorphism in $\mathbf{Smith}_1$, so by Corollary 3.2.23 i is an isomorphism in $\mathbf{Ban}_1$. Therefore $A^\beta = A^\beta_+ - A^\beta_+$. The map $\text{ev}(u) : A^\beta \to \mathbb{R}$ is positive because if $\phi \in A^\beta_+$, we have $\text{ev}(u)(\phi) = \phi(u) \geq 0$ because $u \in A_+$. Suppose $A^\beta \neq \{0\}$, and therefore $A^\beta_+ \neq \{0\}$. Then there exists $\phi \in A^\beta_+ \neq 0$. If $\text{ev}(u)(\phi) = \phi(u) = 0$, Lemma A.5.4 implies $\phi = 0$, contradicting the assumption. We now define $B = A^\beta_+ \cap \text{ev}(u)^{-1}(1)$.

- $\text{absco}(B) \subseteq V$:

  Recall that $V = \text{co}(-B^{\leq 1} \cup B^{\leq 1})$, and $B^{\leq 1} = A^\beta_+ \cap \text{ev}(u)^{-1}((-\infty, 1])$. As $B^{\leq 1}$ contains zero, V is nonempty, and it is convex by definition. If $\alpha\phi_+ - (1-\alpha)\phi_-$ is an element of V, *i.e.* $\alpha \in [0,1]$, $\phi_+, \phi_- \in B^{\leq 1}$, then

  $$-(\alpha\phi_+ - (1-\alpha)\phi_-) = (1-\alpha)\phi_- - \alpha\phi_+,$$

  which is also an element of V. Therefore V is balanced, and so absolutely convex by Lemma A.3.1. Since $B \subseteq B^{\leq 1}$ we have $\text{absco}(B) \subseteq \text{absco}(B^{\leq 1}) \subseteq V$.

- $V \subseteq \text{absco}(B)$:

  As $\text{absco}(B)$ is balanced, $B^{\leq 1} \subseteq \text{absco}(B)$ iff $-B^{\leq 1} \subseteq \text{absco}(B)$, and as it is convex each of these implies $V \subseteq \text{absco}(B)$. Therefore we reduce to showing that $B^{\leq 1} \subseteq \text{absco}(B)$. Let $\phi \in B^{\leq 1}$. If $\phi(u) = 0$, we have $\phi = 0$ and so $\phi \in \text{absco}(B)$ (Lemma A.5.4). If $\phi(u) \neq 0$, and therefore $\phi(u) \in (0,1]$, define $\alpha = \phi(u)$. We have that $\alpha^{-1}\phi$ is in A^β_+ and maps u to 1, hence is an element of B. Therefore $\phi = (1-\alpha)0 + \alpha(\alpha^{-1}\phi)$ is an element of $\text{absco}(B)$.

As we know that V is radially compact, we have $\mathrm{absco}(B)$ is radially compact, so we have a pre-base-norm space. We also showed already that A_+^β is complete, and therefore closed in $\|\text{-}\|_V$, so we have a base-norm space. Since $F = A^\beta$ is complete in $\|\text{-}\|_V$, we have a Banach base-norm space. We have finished with the objects part of the functor.

Now we show that for $f : (A, \mathcal{T}, A_+, u) \to (B, \mathcal{S}, B_+, v)$ a continuous subunital map $G^\beta(f)$ is a trace-reducing map, which is trace-preserving if f is unital. Since $G^\beta(f) = f^\beta$, it is a linear map (Proposition 3.2.20, and it is positive by the positivity argument in Proposition 2.5.2. Additonally, the proofs that $G^\beta(f)$ is trace-reducing or trace-preserving when f is subunital or unital respectively are similar to the proofs for G in Proposition 2.5.2. Preservation of identity maps and composition follows from the identity map laws and associativity of composition of continuous linear maps in the usual way. □

We can also define $F^\sigma : \mathbf{BBNS} \to \mathbf{SOUS}^{\mathrm{op}}$. We give it almost the same definitions as $F : \mathbf{BBNS} \to \mathbf{BOUS}^{\mathrm{op}}$:

$$F^\sigma(E, E_+, \tau) = (E^\sigma, \sigma(E^\sigma, E)_{[-\tau,\tau]}, E_+^\sigma, \tau)$$
$$F^\sigma(f)(b) = b \circ f,$$

where $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ is a trace-reducing map and $b \in F^\sigma$.

Theorem 3.4.5. *F^σ is a functor from $\mathbf{BBNS} \to \mathbf{SOUS}^{\mathrm{op}}$. The restriction of the adjoint equivalence defined by σ and β makes an adjoint equivalence $\mathbf{BBNS} \simeq \mathbf{SOUS}^{\mathrm{op}}$ and $\mathbf{BBNS}_{\leq 1} \simeq \mathbf{SOUS}^{\mathrm{op}}_{\leq 1}$.*

Proof. We have that $F^\sigma(E, E_+, \tau)$ is a Banach order-unit space by Proposition 2.4.17, and that $[-\tau, \tau]$ is the unit ball in the usual dual norm. Therefore $\sigma(E^\sigma, E)_{[-\tau,\tau]}$ is a Smith topology on $F^\sigma(E, E_+, \tau)$ (Proposition 3.2.9). All we need to show that $F^\sigma(E, E_+, \tau)$ is a Smith order-unit space is to show that E_+ is closed. We know that it is closed in $\sigma(E^\sigma, E)$ because it is a dual cone, and therefore a polar. Therefore it is closed in the finer $\sigma(E^\sigma, E)_{[-\tau,\tau]}$-topology.

In Proposition 2.5.2 it is shown that if $f : (E, E_+, \tau) \to (F, F_+, \sigma)$ is trace-reducing, then $F(f)$ is subunital, and if f is trace-preserving, $F(f)$ is unital, as well as F preserving identity maps and composition. Therefore to show F is functor we only need to show that if f is trace-reducing, $F^\sigma(f)$ is continuous in the Smith topologies. This has already been shown for f a bounded linear map in Proposition 3.2.21, so we only need to use Proposition 2.2.12, that trace-reducing implies bounded.

We consider the usual

$$\eta_E : E \to G^\beta(F^\sigma(E)) \qquad \epsilon_A : A \to F^\sigma(G^\beta(A))$$
$$\eta_E(x)(a) = a(x) \qquad \epsilon_A(a)(\phi) = \phi(a).$$

By Theorem 3.2.22 these are linear homeomorphisms of the underlying topological vector spaces, and the diagrams required for an adjoint equivalence commute. Therefore we only need to show that η_E is an isomorphism in $\mathbf{BBNS}$ and ϵ_A an isomorphism in $\mathbf{SOUS}$.

If $x \in E_+$, then for all $a \in E^\sigma_+$, $\eta_E(x)(a) = a(x) \geq 0$, so $\eta_E(x) \in E^{\sigma\beta}_+$. If, on the other hand, if $\phi \in G^\beta(F^\sigma(E))_+$, *i.e.* $E^{\sigma\beta}_+$, then by bijectivity of η_E there exists an $x \in E_+$ such that $\eta_E(x) = \phi$. So for all $a \in E^\sigma_+$ there $a(x) = \eta_E(x)(a) \geq 0$, so x is in the dual cone of E^σ under the duality $(E, E^\sigma, \langle\text{-},\text{-}\rangle)$. As E_+ is closed, by the definition of a base-norm space, Lemma 0.3.15 shows that $x \in E_+$. Therefore η_E and its inverse are both positive.

The proof that ϵ_A and its inverse are positive is similar, using the fact that A_+ is required to be closed in the Smith topology as part of the definition of a Smith order-unit space.

The proofs of trace-preservation and unitality are similar to Theorem 3.3.7, so are omitted. □

3.4.1 Relationship to Convex Sets

In this subsection we show that BAff produces a Smith order-unit space and that **CBConv**, and therefore **BOUS**, forms a reflective subcategory of both $\mathcal{EM}(\mathcal{D})$ and $\mathcal{EM}(\mathcal{D}_\infty)$ via the comparison functors (2.3) and (2.5). Pumplün's result in [106, Theorem 3.3], although "base-norm space" in that reference refers to a pre-base-norm space, so the result is not quite the same.

To redefine BAff, we first define $\eta_X : X \to \mathrm{Stat}(\mathrm{BAff}(X))$ as

$$\eta_X(x)(a) = a(x).$$

Lemma 3.4.6. *For all (X, α_X) in $\mathcal{EM}(\mathcal{D})$ and $x \in X$, $\eta_X(x)$ is a state on* $\mathrm{BAff}(X)$.

Proof. To show that $\eta_X(x)$ is linear, consider $a, b \in \mathrm{BAff}(X)$. Then

$$\eta_X(x)(a + b) = (a + b)(x) = a(x) + b(x) = \eta_X(x)(a) + \eta_X(x)(b).$$

And if $\alpha \in \mathbb{R}$,

$$\eta_X(x)(\alpha a) = (\alpha a)(x) = \alpha a(x) = \alpha\eta_X(x)(a).$$

For positivity, let $a \in \mathrm{BAff}(X)_+$. Then

$$\eta_X(x)(a) = a(x) \geq 0,$$

by the definition of $\mathrm{BAff}(X)_+$. For unitality, $\eta_X(x)(u) = u(x) = 1$ by the definition of u. □

We have that $\eta_X(X) \subseteq \mathrm{BAff}(X)^*$ because all states are continuous (Proposition 1.2.8). Therefore $\mathrm{span}(\eta_X(X))$ is a subspace of $\mathrm{BAff}(X)^*$.

Lemma 3.4.7. *The set $\eta_X(X) \subseteq \mathrm{BAff}(X)^*$ separates the points of* $\mathrm{BAff}(X)$*, therefore the topology $\sigma(\mathrm{BAff}(X), \mathrm{span}(\eta_X(X)))$ is a Hausdorff locally convex topology.*

Proof. Let $a, b \in \mathrm{BAff}(X)$. If for all $\phi \in \mathrm{span}(\eta_X(X))$, then for all $x \in X$ we have $\eta_X(x)(a) = \eta_X(x)(b)$, *i.e.* $a(x) = b(x)$, so $a = b$. Therefore $\sigma(\mathrm{BAff}(X), \eta(X)) = \sigma(\mathrm{BAff}(X), \mathrm{span}(\eta_X(X))$ is a Hausdorff locally convex topology. □

We can now show that $\mathrm{BAff}(X)$ is a Smith space.

Proposition 3.4.8. *The interval* $[0,1]_{\mathrm{BAff}(X)}$ *is* $\sigma(\mathrm{BAff}(X), \eta_X(X))$ *compact, so* $\mathrm{BAff}(X)$ *is a Smith space.*

Proof. Each element $a \in [0,1]_{\mathrm{BAff}(X)}$ is a function from $X \to [0,1]$, or an element of $[0,1]^X$. By Tychonoff's theorem, $[0,1]^X$ is a compact Hausdorff space, so if we show that $[0,1]_{\mathrm{BAff}(X)}$ is closed in $[0,1]^X$, with its standard topology, and that the topology agrees with the $\sigma(\mathrm{BAff}(X), \mathrm{span}(\eta_X(X)))$ topology, we have shown that $[0,1]_{\mathrm{BAff}(X)}$ is compact.

First we show that every $\sigma(\mathrm{BAff}(X), \eta_X(X))$ neighbourhood of a point $a \in [0,1]_{\mathrm{BAff}(X)}$ is a neighbourhood in subspace topology from $[0,1]^X$. A base of neighbourhoods is defined by sets defined as follows. Given a finite set I, and finite sequences $(x_i)_{i\in I}$, $x_i \in X$, and $(\epsilon_i)_{i\in I}$, $\epsilon_i \in \mathbb{R}_{>0}$, we take

$$N_{a,(x_i),(\epsilon_i)} = \bigcap_{i\in I} N_{a,x_i,\epsilon_i} = \bigcap_{i\in I} \{b \in \mathrm{BAff}(X) \mid |a(x_i) - b(x_i)| < \epsilon_i\}.$$

Sets of this form make up a base of a-neighbourhoods in $\sigma(\mathrm{BAff}(X), \eta_X(X))$. Now, $|a(x_i) - b(x_i)| < \epsilon_i$ iff $b(x_i) \in (a(x_i) - \epsilon_i, a(x_i) + \epsilon_i)$, and

$$\{b \in [0,1]^X \mid b(x_i) \in (a(x_i) - \epsilon_i, a(x_i) + \epsilon_i)\} = \pi_{x_i}^{-1}((a(x_i) - \epsilon_i, a(x_i) + \epsilon_i))$$

is an open set in the product topology, so we have shown that the product topology is finer than $\sigma(\mathrm{BAff}(X), \eta_X(X))$.

Now, let I be a finite set, $(U_i)_{i\in I}$ a finite sequence of open subsets of $\mathbb{R}$, and $(x_i)_{i\in I}$ a finite sequence of elements of X such that $a(x_i) \in U_i$. Then

$$\bigcap_{i\in I} \pi_{x_i}^{-1}(U_i)$$

is an open neighbourhood of a in the product topology, and the family of sets of this form make a neighbourhood base for a. As each U_i is open, we can pick an $(\epsilon_i)_{i\in I}$ such that $(a(x) - \epsilon_i, a(x) + \epsilon_i) \subseteq U_i$ for all $i \in I$. Then $N_{a,x_i,\epsilon_i} \subseteq \pi_{x_i}^{-1}(U_i)$, so the $\sigma(\mathrm{BAff}(X), \eta_X(X))$ topology is finer than the product topology. Combining this with the previous paragraph proves that they are both the same.

We can now move on to showing that $[0,1]_{\mathrm{BAff}(X)} \subseteq [0,1]^X$ is closed. Let $(a_i)_{i\in I}$ be a net in $[0,1]_{\mathrm{BAff}(X)}$ converging to $a \in [0,1]^X$. Since $a \in [0,1]^X$ is bounded, we only need to show that it is $\mathcal{D}$-affine. Let $\phi \in \mathcal{D}$, and we want to show that $\sum_{x\in X} \phi(x)a(x) = a(\alpha_X(\phi))$. We have that $\Phi(a_i) = \sum_{x\in X} \phi(x)a_i(x) = a_i(\alpha_X(\phi))$ for all $i \in I$. Since convergence in the product topology is pointwise, we have that $a_i(\alpha_X(\phi)) \to a(\alpha_X(\phi))$. Since the sum is finite, and addition and scalar multiplication are continuous, $\Phi(a_i) \to \Phi(a)$. As the topology is Hausdorff, we have that $\Phi(a) = a(\alpha_X(\phi))$, *i.e.* a is $\mathcal{D}$-affine.

We then apply Proposition 3.4.1 to conclude that

$$(\mathrm{BAff}(X), \sigma(\mathrm{BAff}(X), \eta_X(X))_{[-u,u]}, \mathrm{BAff}(X)_+, u)$$

is a Smith order-unit space. □

Proposition 3.4.9. *For all maps* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *in* $\mathcal{EM}(\mathcal{D})$*,* $\mathrm{BAff}(f)$ *is a continuous map of Smith spaces, so* BAff *is a functor* $\mathcal{EM}(\mathcal{D}) \to \mathbf{SOUS}^{\mathrm{op}}$*.*

Proof. We show that $\mathrm{BAff}(f) : \mathrm{BAff}(Y) \to \mathrm{BAff}(X)$ is continuous from the topology $\sigma(\mathrm{BAff}(Y), \eta_Y(Y))$ to $\sigma(\mathrm{BAff}(X), \eta_X(X))$. Therefore $[0,1]_{\mathrm{BAff}(f)}$ is continuous, so by Theorem 3.4.3 $\mathrm{BAff}(f)$ is continuous in the Smith space topologies. Then BAff is a functor $\mathcal{EM}(\mathcal{D}) \to \mathbf{SOUS}^{\mathrm{op}}$ by the same proof that in Theorem 2.4.16 for $\mathrm{BAff} : \mathcal{EM}(\mathcal{D}) \to \mathbf{BOUS}^{\mathrm{op}}$.

We show that if $M_{x,\epsilon}$ (where $x \in X$ and $\epsilon \in \mathbb{R}_{>0}$) is a subbasic 0-neighbourhood in $\mathrm{BAff}(X)$, then $N_{f(x),\epsilon}$ is a subbasic 0-neighbourhood in $\mathrm{BAff}(Y)$ such that $N_{f(x),\epsilon} \subseteq \mathrm{BAff}(f)^{-1}(M_{x,\epsilon})$, which implies continuity by the fact that $\mathrm{BAff}(f)^{-1}$ preserves intersections.

So let $b \in N_{f(x),\epsilon}$, *i.e.* $|b(f(x))| < \epsilon$. Then we want to show that $\mathrm{BAff}(f)(b) \in M_{x,\epsilon}$. We have that $\mathrm{BAff}(f)(b)(x) = b(f(x))$. So $|b(f(x))| < \epsilon$ implies that $|\mathrm{BAff}(f)(b)(x)| < \epsilon$, so $\mathrm{BAff}(f)(b) \in M_{x,\epsilon}$. □

Recall the natural isomorphisms $\rho : F \Rightarrow \mathrm{BAff} \circ B$ from Theorem 2.4.18, where BAff are the functors to $\mathbf{BOUS}^{\mathrm{op}}$. We have seen above how to make BAff into a functor to $\mathbf{SOUS}^{\mathrm{op}}$, and we have seen in Theorem 3.4.5 how to make F into a functor $F^\sigma : \mathbf{BBNS} \to \mathbf{SOUS}^{\mathrm{op}}$ by giving $F(E)$ a Smith topology.

Proposition 3.4.10. ρ *is a natural isomorphism* $F^\sigma \Rightarrow \mathrm{BAff} \circ B^{\mathcal{D}}$ *and* $F^\sigma \Rightarrow \mathrm{BAff} \circ B^{\mathcal{D}_\infty}$*.*

Proof. We show this for $B^{\mathcal{D}}$, and then it follows for $B^{\mathcal{D}_\infty}$ by Lemma 2.4.14.

We only need to show that ρ_E is a homeomorphism, *i.e.* it is continuous and open. To do this, we show that the image of the $\sigma(F^\sigma, F)$ topology on F^σ is the $\sigma(\mathrm{BAff}(B^{\mathcal{D}}(E)), \eta_{B^{\mathcal{D}}(E)}(B^{\mathcal{D}}(E)))$ topology. First observe that $\sigma(F^\sigma, F) = \sigma(F^\sigma, B^{\mathcal{D}}(E))$ as $\mathrm{span}(B^{\mathcal{D}}(E)) = F$. So the family of sets $N_{x,\epsilon}$ where $x \in B^{\mathcal{D}}(E)$ and $\epsilon \in \mathbb{R}_{>0}$ is a subbasis for $\sigma(F^\sigma, F)$. We have

$$\begin{aligned}
\rho_E(N_{x,\epsilon}) &= \{\rho_E(a) \mid a \in N_{x,\epsilon}\} = \{\rho_E(a) \mid a \in E^\sigma \text{ and } |a(x)| < \epsilon\} \\
&= \{\rho_E(a) \mid a \in E^\sigma \text{ and } |\rho_E(a)(x)| < \epsilon\} \\
&= \{a \in \mathrm{BAff}(B^{\mathcal{D}}(E)) \mid |a(x)| < \epsilon\} \\
&= \{a \in \mathrm{BAff}(B^{\mathcal{D}}(E)) \mid |\eta_{B^{\mathcal{D}}(E)}(x)(a)| < \epsilon\} = M_{x,\epsilon}
\end{aligned}$$

which is a subbasic neighbourhood for $\sigma(\mathrm{BAff}(B^{\mathcal{D}}(E)), \eta_{B^{\mathcal{D}}(E)}(B^{\mathcal{D}}(E)))$. Conversely, if we start with a subbasic neighbourhood for the locally convex topology $\sigma(\mathrm{BAff}(B^{\mathcal{D}}(E)), \eta_{B^{\mathcal{D}}(E)}(B^{\mathcal{D}}(E)))$, it is always the image of a subbasic neighbourhood for $\sigma(E^\sigma, B^{\mathcal{D}}(E))$, so the two topologies are the same. Since the unit ball of E^σ is mapped to the unit ball of $\mathrm{BAff}(B^{\mathcal{D}}(E))$ by ρ_E, it is a linear homeomorphism of the Smith topologies as well.

The proof of naturality then carries over from Theorem 2.4.18. □

We define the functor CStat : $\mathbf{SOUS}^{\mathrm{op}} \to \mathcal{EM}(\mathcal{D}_\infty)$ to be $B^{\mathcal{D}_\infty} \circ G^\beta$, we reuse the name for CStat : $\mathbf{SOUS}^{\mathrm{op}} \to \mathcal{EM}(\mathcal{D})$, defined as $B^{\mathcal{D}} \circ G^\beta$, and CStat : $\mathbf{SOUS}^{\mathrm{op}} \to \mathbf{BConv}$, defined as $B \circ G^\beta$, as it will be clear from context which is meant. It is also clear that $\mathrm{CStat}(A, A_+, u)$ consists of continuous positive linear functionals $\phi : A \to \mathbb{R}$ such that $\phi(u) = 1$, *i.e.* continuous states, hence the name.

We define $\eta_X : X \to \mathrm{CStat}(\mathrm{BAff}(X))$ as

$$\eta_X(x)(a) = a(x),$$

where (X, α_X) can be a $\mathcal{D}$-algebra or a $\mathcal{D}_\infty$-algebra, $x \in X$ and $a \in \mathrm{BAff}(X)$. We also define $\epsilon_A : A \to \mathrm{BAff}(\mathrm{CStat}(A))$.

$$\epsilon_A(a)(\phi) = \phi(a),$$

where A is a Smith order-unit space, $a \in A$, and $\phi \in \mathrm{CStat}(A)$.

For ease of notation, from now on we use $\mathcal{D}_?$ to refer to either $\mathcal{D}_\infty$ or $\mathcal{D}$ in cases where the proof works both ways.

Theorem 3.4.11. BAff *is a left adjoint to* CStat*, whether we are using* $\mathcal{EM}(\mathcal{D}_\infty)$ *or* $\mathcal{EM}(D)$*, with* η *and* ϵ *being given by the definitions above, and* ϵ *is a natural isomorphism.*

Proof. We first show that η and ϵ are natural transformations, then that they satisfy the triangle identities necessary for an adjunction (from Theorem 0.4.1 (v)).

We first need to show that $\eta_X(x) \in \mathrm{CStat}(\mathrm{BAff}(X))$. It is a state by Lemma 3.4.6. We know from Proposition 3.4.8 that the Smith topology for $\mathrm{BAff}(X)$ is the Smithization of $\sigma(\mathrm{BAff}(X), \eta_X)$ with respect to $[-u, u]$. By definition $\eta_X(x)$ is $\sigma(\mathrm{BAff}(X), \eta_X(x))$-continuous, so $\eta_X(x)|_{[-u,u]}$ is continuous, so by Proposition 3.2.15 $\eta_X(x)$ is a continuous state.

We want to show that η_X is an $\mathcal{EM}(\mathcal{D}_?)$-morphism, where $\mathcal{D}_?$ means either $\mathcal{D}$ or $\mathcal{D}_\infty$, as appropriate. To show this we need to show

$$\begin{array}{ccc} \mathcal{D}_?(X) & \xrightarrow{\mathcal{D}_?(\eta_X)} & \mathcal{D}_?(\mathrm{CStat}(\mathrm{BAff}(X))) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_{\mathrm{CStat}(\mathrm{BAff}(X))}} \\ X & \xrightarrow[\eta_X]{} & \mathrm{CStat}(\mathrm{BAff}(X)) \end{array}$$

commutes. So let $\psi \in \mathcal{D}_?(X)$, and $a \in \mathrm{BAff}(X)$. For the lower left path we get

$$\eta_X(\alpha_X(\psi))(a) = a(\alpha_X(\psi)) = \sum_{x \in X} \psi(x) a(x).$$

For the upper right path we get

$$\begin{aligned}\alpha_{\mathrm{CStat}(\mathrm{BAff}(X))}(\mathcal{D}_?(\eta_X)(\psi))(a) &= \sum_{\phi \in \mathrm{CStat}(\mathrm{BAff}(X))} \mathcal{D}_?(\eta_X)(\psi)(\phi) \cdot \phi(a) \\ &= \sum_{\phi \in \mathrm{CStat}(\mathrm{BAff}(X))} \left(\sum_{x \in \eta_X^{-1}(\phi)} \psi(x) \right) \phi(a).\end{aligned}$$

If $\eta_X^{-1}(\phi) = \emptyset$ then the inner sum is zero and so can be omitted, so we can restrict the outer sum to be over $\eta_X(X)$. We also pick an x_ϕ for each $\phi \in \eta_X(X)$, such that $\eta_X(x_\phi) = \phi$. Resuming,

$$\begin{aligned} &= \sum_{\phi \in \eta_X(X)} \left(\sum_{x \in \eta_X^{-1}(\phi)} \psi(x) \right) \eta_X(x_\phi)(a) \\ &= \sum_{x \in X} \psi(x) \cdot \eta_X(x_{\eta_X(x)})(a) \\ &= \sum_{x \in X} \psi(x) \eta_X(x)(a) && \text{because } \eta_X(x_{\eta_X(x)}) = \eta_X(x) \\ &= \sum_{x \in X} \psi(x) a(x).\end{aligned}$$

Therefore $\eta_X(x)$ is an $\mathcal{EM}(\mathcal{D}_?)$-morphism.

The naturality diagram of η is

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathrm{CStat}(\mathrm{BAff}(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{CStat}(\mathrm{BAff}(f))} \\ Y & \xrightarrow[\eta_Y]{} & \mathrm{CStat}(\mathrm{BAff}(Y)). \end{array}$$

For $X, Y \in \mathcal{EM}(\mathcal{D}_?)$, $x \in X$ and $b \in \mathrm{BAff}(Y)$, then for the lower left path we have

$$\eta_Y(f(x))(b) = b(f(x)),$$

while for the upper right path, we have

$$\mathrm{CStat}(\mathrm{BAff}(f))(\eta_X(x))(b) = \eta_X(x)(\mathrm{BAff}(f)(b)) = \mathrm{BAff}(f)(b)(x) = b(f(x)),$$

so the diagram commutes, and so η is natural.

We now show that ϵ is a natural ismorphism $\mathrm{Id} \circ \mathrm{BAff} \circ \mathrm{CStat}$. To avoid confusion, we temporarily use the notation ϵ' for the counit from Theorem 3.4.5. We have that $\rho G^\beta \circ \epsilon'$ is a natural isomorphism $\mathrm{Id} \Rightarrow \mathrm{BAff} \circ B^{\mathcal{D}_?} \circ G^\beta$, and $\mathrm{BAff} \circ B^{\mathcal{D}_?} \circ G^\beta = \mathrm{BAff} \circ \mathrm{CStat}$. Therefore $\rho G^\beta \circ \epsilon'$ has the type that we want ϵ to have, and it is a natural isomorphism by Theorem 3.4.5 and Proposition

3.4.10. Now, if A is a Smith order-unit space, $a \in A$, and $\phi \in \mathrm{CStat}(A)$, we have

$$\rho_{G^\beta(A)}(\epsilon'_A(a))(\phi) = \epsilon'_A(a)(\phi) = \phi(a) = \epsilon_A(a)(\phi),$$

so $\epsilon = \rho G^\beta \circ \epsilon'$ and is therefore a natural isomorphism.

The diagrams for an adjunction $\mathrm{BAff} \dashv \mathrm{CStat}$ are

$$\begin{array}{ccc} \mathrm{CStat}(A) & \xrightarrow{\eta_{\mathrm{CStat}(A)}} & \mathrm{CStat}(\mathrm{BAff}(\mathrm{CStat}(A))) \\ & \searrow^{\mathrm{id}_{\mathrm{CStat}(A)}} & \downarrow \mathrm{CStat}(\epsilon_A) \\ & & \mathrm{CStat}(A) \end{array}$$

$$\begin{array}{ccc} \mathrm{BAff}(X) & \xleftarrow{\mathrm{BAff}(\eta_X)} & \mathrm{BAff}(\mathrm{CStat}(\mathrm{BAff}(X))) \\ & \nwarrow^{\mathrm{id}_{\mathrm{BAff}(X)}} & \uparrow \epsilon_{\mathrm{BAff}(X)} \\ & & \mathrm{BAff}(X) \end{array}$$

where A is a Smith order-unit space, and X an Eilenberg-Moore algebra of $\mathcal{D}_?$.

Let $\phi \in \mathrm{CStat}(A)$ and $a \in A$. Then

$$\mathrm{CStat}(\epsilon_A)(\eta_{\mathrm{CStat}(A)}(\phi))(a) = \eta_{\mathrm{CStat}(A)}(\phi)(\epsilon_A(a)) = \epsilon_A(a)(\phi) = \phi(a),$$

so $\mathrm{CStat}(\epsilon_A)(\eta_{\mathrm{CStat}(A)}(\phi)) = \phi$ and the top diagram commutes. Now, if $a \in \mathrm{BAff}(X)$ and $x \in X$,

$$\mathrm{BAff}(\eta_X)(\epsilon_{\mathrm{BAff}(X)}(a))(x) = \epsilon_{\mathrm{BAff}(X)}(a)(\eta_X(x)) = \eta_X(x)(a) = a(x),$$

so the bottom diagram commutes as well. □

As $B : \mathbf{BBNS} \to \mathbf{CBConv}$ is an equivalence (Proposition 2.4.13), Theorem 0.4.3 implies the existence of a functor $\mathrm{Emb} : \mathbf{CBConv} \to \mathbf{BBNS}$ such that there exist a unit and counit making B and Emb part of an adjoint equivalence, and we have both $B \dashv \mathrm{Emb}$ and $\mathrm{Emb} \dashv B$.

Corollary 3.4.12. *The functors* $B^{\mathcal{D}_?} : \mathbf{BBNS} \to \mathcal{EM}(\mathcal{D}_?)$ *and the comparison functor* $\mathbf{CBConv} \to \mathcal{EM}(\mathcal{D}_?)$ *have left adjoints, making* $\mathbf{BBNS}$ *and* $\mathbf{CBConv}$ *full reflective subcategories of* $\mathcal{EM}(\mathcal{D}_?)$*.*

Proof. We have that $\mathrm{BAff} \dashv \mathrm{CStat} = B^{\mathcal{D}_?} \circ G^\beta$, and $G^\beta \dashv F^\sigma$ by Theorems 3.4.11 and 3.4.5 respectively. By composing the adjunctions, we have that $G^\beta \circ \mathrm{BAff} \dashv B^{\mathcal{D}_?} \circ G^\beta \circ F^\sigma$. Applying Theorem 3.4.5 again, we have $B^{\mathcal{D}_?} \circ G^\beta \circ F^\sigma \cong B^{\mathcal{D}_?}$, so $G^\beta \circ \mathrm{BAff} \dashv B^{\mathcal{D}_?}$.

The comparison functor $K : \mathbf{CBConv} \to \mathcal{EM}(\mathcal{D}_?)$ is $B^{\mathcal{D}_?} \circ \mathrm{Emb}$. We can compose the previous adjunction with $B \dashv \mathrm{Emb}$ to get $B \circ G^\beta \circ \mathrm{BAff} \dashv B^{\mathcal{D}_?} \circ \mathrm{Emb}$. Using our definitions, this is $\mathrm{CStat} \circ \mathrm{BAff} \dashv K$. □

We now have three sides of the square (3.2). We define CStat : $\mathbf{CEMod}^{\mathrm{op}} \to \mathbf{CBConv}$ to be $B(G^{\beta}(A))$ for all compact effect modules of the form $[0,1]_A$. As $[0,1]_{\text{-}}$ is an equivalence, this can be extended to all of $\mathbf{CEMod}^{\mathrm{op}}$, all such extensions being naturally isomorphic. This implies $\mathrm{CStat} \circ [0,1]_{\text{-}} = B \circ G^{\beta}$. We also define BAff(-, [0, 1]) as

$$\begin{aligned} \mathrm{BAff}(X,[0,1]) &= \{a \in \mathrm{BAff}(X) \mid \forall x \in X.0 \le a(x) \le 1\} \\ \mathrm{BAff}(f,[0,1])(b) &= b \circ f \end{aligned}$$

where X, Y are objects of **CBConv**, $f : X \to Y$ is an affine map and $b \in \mathrm{BAff}(Y)$. It is clear that $\mathrm{BAff}(\text{-},[0,1]) = [0,1]_{\text{-}} \circ \mathrm{BAff}$, and therefore a functor.

Theorem 3.4.13. *In* (3.2) *we have* $\mathrm{BAff}(\text{-},[0,1]) \circ B \cong [0,1]_{\text{-}} \circ F^{\sigma}$ *and that* BAff(-, [0, 1]) *and* CStat *define an equivalence between* **CBConv** *and* $\mathbf{CEMod}^{\mathrm{op}}$.

Proof. We have a natural isomorphism $\rho : F^{\sigma} \Rightarrow \mathrm{BAff} \circ B$ from Proposition 3.4.10. Therefore $[0,1]_{\rho} : [0,1]_{\text{-}} \circ F^{\sigma} \Rightarrow [0,1]_{\text{-}} \circ \mathrm{BAff} \circ B = \mathrm{BAff}(\text{-},[0,1]) \circ B$, which is the required isomorphism.

To prove that BAff(-, [0, 1]) and CStat define an equivalence, we show that $\mathrm{BAff}(\text{-},[0,1]) \circ \mathrm{CStat} \cong \mathrm{Id}_{\mathbf{CEMod}}$ and $\mathrm{CStat} \circ \mathrm{BAff}(\text{-},[0,1]) \cong \mathrm{Id}_{\mathbf{CBConv}}$. We reason as follows

$$\begin{aligned} \mathrm{BAff}(\text{-},[0,1]) \circ B &\cong [0,1]_{\text{-}} \circ F^{\sigma} && \Leftrightarrow \\ \mathrm{BAff}(\text{-},[0,1]) \circ B \circ \mathrm{Emb} &\cong [0,1]_{\text{-}} \circ F^{\sigma} \circ \mathrm{Emb} && \Leftrightarrow \\ \mathrm{BAff}(\text{-},[0,1]) &\cong [0,1]_{\text{-}} \circ F^{\sigma} \circ \mathrm{Emb}, \end{aligned}$$

as Emb is an inverse for B. Similarly, we have

$$\begin{aligned} \mathrm{CStat} \circ [0,1]_{\text{-}} &= B \circ G^{\beta} && \Leftrightarrow \\ \mathrm{CStat} \circ [0,1]_{\text{-}} \circ \mathcal{T} &= B \circ G^{\beta} \circ \mathcal{T} && \Leftrightarrow \\ \mathrm{CStat} &\cong B \circ G^{\sigma} \circ \mathcal{T}, \end{aligned}$$

as $\mathcal{T}$ is an inverse for $[0,1]_{\text{-}}$.

Then we have

$$\begin{aligned} \mathrm{CStat} \circ \mathrm{BAff}(\text{-},[0,1]) &\cong B \circ G^{\beta} \circ \mathcal{T} \circ [0,1]_{\text{-}} \circ F^{\sigma} \circ \mathrm{Emb} \\ &\cong B \circ G^{\beta} \circ F^{\sigma} \circ \mathrm{Emb} && \text{Theorem 3.4.3} \\ &\cong B \circ \mathrm{Emb} && \text{Theorem 3.4.5} \\ &\cong \mathrm{Id}_{\mathbf{CBConv}} && \text{Proposition 2.4.13.} \end{aligned}$$

On the other side, we have

$$\begin{aligned} \mathrm{BAff}(\text{-},[0,1]) \circ \mathrm{CStat} &\cong [0,1]_{\text{-}} \circ F^{\sigma} \circ \mathrm{Emb} \circ B \circ G^{\beta} \circ \mathcal{T} \\ &\cong [0,1]_{\text{-}} \circ F^{\sigma} \circ G^{\beta} \circ \mathcal{T} && \text{Proposition 2.4.13} \\ &\cong [0,1]_{\text{-}} \circ \mathcal{T} && \text{Theorem 3.4.5} \\ &\cong \mathrm{Id}_{\mathbf{CEMod}} && \text{Theorem 3.4.3.} \end{aligned}$$

□

3.5 Universal Enveloping Objects

We can combine the adjunction $F \dashv G$[5] and the adjoint equivalences $F^\sigma \dashv G^\beta$ and $F^\beta \dashv G^\sigma$ to produce adjunctions analogous to the enveloping W*-algebra of a C*-algebra.

By Corollary 3.2.18 each Smith base-norm space $(E, \mathcal{T}, E_+, \tau)$ has an underlying Banach base-norm space $U(E) = (E, E_+, \tau)$, and each Smith order-unit space $(A, \mathcal{T}, A_+, u)$ has an underlying Banach order-unit space $V(A) = (A, A_+, u)$ (in each case the positive cone remains closed because the topology is finer). As the maps in **SBNS** and **SOUS** are maps in **BBNS** and **BOUS**, required to be continuous, we have forgetful functors $U : \mathbf{SBNS} \to \mathbf{BBNS}$ and $V : \mathbf{SOUS} \to \mathbf{BOUS}$. We also see that by definition $G = UG^\sigma$ and $F = V^{\mathrm{op}} F^\sigma$.

Theorem 3.5.1. *The functor $U \cong GF^\beta$, and $V^{\mathrm{op}} \cong FG^\beta$. By composition of adjunctions, $G^\sigma F \dashv U$ and $F^\sigma G \dashv V^{\mathrm{op}}$, or equivalently $V \dashv (F^\sigma G)^{\mathrm{op}}$.*

Proof. From Theorem 3.3.7, we have the isomorphism $\eta : \mathrm{Id}_{\mathbf{SBNS}} \Rightarrow G^\sigma F^\beta$, so $U\eta : U \Rightarrow GF^\beta$ is a natural isomorphism. Analogously, $V^{\mathrm{op}}\epsilon : V^{\mathrm{op}} \to FG^\beta$ is a natural isomorphism, using the ϵ from Theorem 3.4.5. If we compose the adjunctions from the above two theorems and the one from Theorem 2.5.4, we have $G^\sigma F \dashv GF^\beta \cong U$ and $V^{\mathrm{op}} \cong FG^\beta \dashv F^\sigma G$, as required. □

On the underlying Banach spaces, $G^\sigma F$ and $(F^\sigma G)^{\mathrm{op}}$ are both double dualization, taking a space E to E^{**}. Therefore the above theorem for $(F^\sigma G)^{\mathrm{op}}$ is a version of Dauns's adjunction for the universal enveloping W*-algebra of a C*-algebra [25, §3].

We can also consider, for an object $(E, X) \in \mathbf{CCL}$, $U'(X) = (E, X)$ is an object of **CBConv** because compact spaces are complete in their unique uniformity [19, II.4.1 Theorem 1]. Again, as maps in **CCL** are just maps in **CBConv** required to be continuous, U' is a functor $\mathbf{CCL} \to \mathbf{CBConv}$.

Theorem 3.5.2. *The functor* $\mathrm{Stat} \circ V^{\mathrm{op}} \circ \mathrm{BAff}$ *is a left adjoint to U'.*

Proof. We first show that $U' \cong BU\mathrm{Emb}$, as follows. By definition the following diagram commutes

$$\begin{array}{ccc} \mathbf{SBNS} & \xrightarrow{B} & \mathbf{CCL} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathbf{BBNS} & \xrightarrow[B]{} & \mathbf{CBConv}, \end{array}$$

so $BU\mathrm{Emb} = U'B\mathrm{Emb}$. By the definition of Emb, we have $B\mathrm{Emb} \cong \mathrm{Id}_{\mathbf{CCL}}$, so composing this isomorphism with U' gives $U' \cong U'B\mathrm{Emb} = BU\mathrm{Emb}$.

[5] $F : \mathbf{PreBNS} \to \mathbf{BOUS}^{\mathrm{op}}$ and $G : \mathbf{OUS}^{\mathrm{op}} \to \mathbf{PreBNS}$

We can then compose adjunctions as follows

$$\begin{array}{c} \mathbf{CCL} \\ B \uparrow \dashv \downarrow \mathrm{Emb} \\ \mathbf{SBNS} \\ G^\sigma F \uparrow \dashv \downarrow U \\ \mathbf{BBNS} \\ \mathrm{Emb} \uparrow \dashv \downarrow B \\ \mathbf{CBConv}, \end{array}$$

showing that $BG^\sigma F\mathrm{Emb} \dashv U'$. But this is not quite the statement we want, so we rearrange the left hand side a little:

$$\begin{aligned} B \circ G^\sigma \circ F \circ \mathrm{Emb} &= \mathrm{Stat} \circ F \circ \mathrm{Emb} && \text{definition of Stat} \\ &= \mathrm{Stat} \circ V \circ F^\sigma \circ \mathrm{Emb} \\ &\cong \mathrm{Stat} \circ V \circ \mathrm{BAff} && \text{Proposition 3.4.10.} \end{aligned}$$

This gives $\mathrm{Stat} \circ V \circ \mathrm{BAff} \dashv U'$, as required. □

This shows that every $(E, X) \in \mathbf{CBConv}$ has a universal compactification. The construction of it given above is an instance of Semadeni compactification [105, Theorem 4.5].

We can also consider the case of effect modules. We can define $V' : \mathbf{CEMod} \to \mathbf{BEMod}$ by dropping the embedding in a topological vector space. The reason the codomain is **BEMod** and not just **EMod** is Corollary 3.2.18.

Theorem 3.5.3. *The functor* $(\mathrm{BAff}(\text{-}, [0, 1]) \circ U' \circ \mathrm{Stat})^{\mathrm{op}}$ *is a left adjoint to* V'.

Proof. First we show that $V' \cong [0, 1]_{\text{-}} \circ V \circ \mathcal{T}$. Observe that by definition, the diagram

$$\begin{array}{ccc} \mathbf{SOUS} & \xrightarrow{[0,1]_{\text{-}}} & \mathbf{CEMod} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V'} \\ \mathbf{BOUS} & \xrightarrow[{[0,1]_{\text{-}}}]{} & \mathbf{BEMod} \end{array}$$

commutes. We have that $[0, 1]_{\text{-}} \circ \mathcal{T} \cong \mathrm{Id}_{\mathbf{CEMod}}$ by Theorem 3.4.3, so $V' \cong V' \circ [0, 1]_{\text{-}} \circ \mathcal{T} = [0, 1]_{\text{-}} \circ V \circ \mathcal{T}$.

We can then compose adjunctions as follows

$$\begin{array}{c} \mathbf{CEMod} \\ {\scriptstyle [0,1]_{\text{-}}}\ \uparrow \dashv \downarrow\ {\scriptstyle \mathcal{T}} \\ \mathbf{SOUS} \\ {\scriptstyle (F^{\sigma}G)^{\mathrm{op}}}\ \uparrow \dashv \downarrow\ {\scriptstyle V} \\ \mathbf{BOUS} \\ {\scriptstyle \mathcal{T}}\ \uparrow \dashv \downarrow\ {\scriptstyle [0,1]_{\text{-}}} \\ \mathbf{CEMod}, \end{array}$$

giving us $[0,1]_{\text{-}} \circ (F^{\sigma}G)^{\mathrm{op}} \circ \mathcal{T} \dashv [0,1]_{\text{-}} \circ V \circ \mathcal{T} \cong V'$. We need to adjust the left hand side a bit:

$$\begin{aligned} &[0,1]_{\text{-}} \circ (F^{\sigma}G)^{\mathrm{op}} \circ \mathcal{T} \\ &\cong [0,1]_{\text{-}} \circ (\mathrm{BAff} \circ B \circ G)^{\mathrm{op}} \circ \mathcal{T} && \text{Proposition 3.4.10} \\ &= (\mathrm{BAff}(\text{-},[0,1]) \circ B \circ G)^{\mathrm{op}} \circ \mathcal{T} && \text{definition of } \mathrm{BAff}(\text{-},[0,1]) \\ &= (\mathrm{BAff}(\text{-},[0,1]) \circ B \circ U \circ G^{\sigma})^{\mathrm{op}} \circ \mathcal{T} \\ &= (\mathrm{BAff}(\text{-},[0,1]) \circ U' \circ B \circ G^{\sigma})^{\mathrm{op}} \circ \mathcal{T} && \text{see Theorem 3.5.2} \\ &\cong (\mathrm{BAff}(\text{-},[0,1]) \circ U' \circ \mathrm{Stat})^{\mathrm{op}}, \end{aligned}$$

the last isomorphism arising from the definition of $\mathrm{Stat} : \mathbf{BEMod} \to \mathbf{CCL}$ and $(\mathcal{T}, [0,1]_{\text{-}})$ forming an equivalence between **BEMod** and **BOUS**. □

3.6 Relationship to C* and W*-algebras

We have $\mathbf{C^*Alg}_{\mathrm{PU}}$ is a full subcategory of **BOUS** via the functor SA (Proposition 1.2.10, so we can produce the following state-effect triangle:

$$\begin{array}{ccc} \mathbf{BEMod}^{\mathrm{op}} & \underset{\mathrm{CAff}(\text{-},[0,1])}{\overset{\mathrm{Stat}}{\rightleftarrows}} & \mathbf{CCL} \\ & {\scriptstyle [0,1]_{\text{-}}}\nwarrow \quad \nearrow{\scriptstyle \mathrm{Stat}} & \\ & \mathbf{C^*Alg}_{\mathrm{PU}}^{\mathrm{op}}, & \end{array}$$

The top line is an adjoint equivalence by Theorem 3.3.9. As $\mathrm{Stat} = \mathrm{Stat} \circ [0,1]_{\text{-}}$ by definition, and

$$\mathrm{CAff}(\text{-},[0,1]) \circ \mathrm{Stat} = \mathrm{CAff}(\text{-},[0,1]) \circ B \circ G^{\sigma} \cong [0,1]_{\text{-}} \circ F^{\beta} \circ G^{\sigma} \cong [0,1]_{\text{-}},$$

the two isomorphisms being from Theorems 3.3.9 and 3.3.7, so the triangle commutes up to isomorphism[6].

[6] We implicitly use the definitions $[0,1]_{\text{-}} \circ \mathrm{SA} = [0,1]_{\text{-}}$ and $\mathrm{Stat} \circ \mathrm{SA} = \mathrm{Stat}$.

The state-and-effect triangle above summarizes how given a quantum program in $\mathbf{C^*Alg}_{\mathrm{PU}}$, one can consider its state transformer semantics in **CCL**, which is also known as the "Schrödinger picture", and also its predicate transformer semantics in $\mathbf{BEMod}^{\mathrm{op}}$, also known as the "Heisenberg picture", and these are equivalent, with no further healthiness conditions. This triangle appeared first in [45], with $\mathcal{EM}(\mathcal{R})$ instead of **CCL**, but we will see in the next chapter that this is equivalent.

State-and-effect triangles themselves appear in [56] as a categorical representation of the duality between states and effects, and the relationship to the Schrödinger and Heisenberg picture is also discussed in a more general setting in [61, 60].

The above implies that Stat : $\mathbf{C^*Alg}_{\mathrm{PU}} \to \mathbf{CCL}$ is full and faithful. As an aside, we note that Alfsen, Hanche-Olsen and Shultz have characterized the essential image of Stat [5, Corollary 8.6]. We do not give the characterization here as it involves many further definitions. We note that it is also possible to produce a characterization of the spaces of *pure* states of C*-algebras[82, §I.3.9], which is closer to what happens in Gelfand duality.

Since there are PU-maps that are not completely positive, Stat is not a full functor when restricted to $\mathbf{C^*Alg}_{\mathrm{CPU}}$, the category of C*-algebras with completely positive unital maps. In fact, whether a map is completely positive or not depends on the orientation (in the sense of [5]) and cannot be defined purely from the **CCL** structure of the state space. This can be seen by the fact that the transpose map, the archetypal positive but not completely positive map, is self-inverse, and hence an isomorphism as a PU map, and so by the above result defines an isomorphism in **CCL** on the state space.

We now move on to W*-algebras. As we saw in Theorem 2.3.1, the self-adjoint part of the predual A_* of a W*-algebra A can be equipped with the structure of a base-norm space such that the dual of A_* is the self-adjoint part of A, where the base is the normal state space.

A morphism of W*-algebras, whether it is positive, completely positive, or a *-homomorphism, is said to be *normal* if it is continuous with respect to the weak-* topologies arising from the preduals. We define $\mathbf{W^*Alg}_{\mathrm{PU}}$ to be the category with W*-algebras as objects and normal PU-maps as morphisms. Similarly, $\mathbf{W^*Alg}$ has normal *-homomorphisms, and $\mathbf{W^*Alg}_{\mathrm{P}\leq 1}$ has normal positive subunital maps. Similar to the C*-algebraic case, we can restrict to full subcategories on commutative W*-algebras, which we call $\mathbf{CW^*Alg}$ and $\mathbf{CW^*Alg}_{\mathrm{PU}}$.

We can define SA, extending the definition from chapter 1 to $\mathbf{W^*Alg}_{\mathrm{PU}}$ as follows.

Lemma 3.6.1. *For any W*-algebra A, the adjoint operation* -* : $A \to \overline{A}$ *is* $\sigma(A, A_*)_b$*-continuous. Therefore* SA(A) *is a* $\sigma(A, A_*)_b$*-closed subspace, and therefore a Smith space. The set of positive operators is* $\sigma(A, A_*)$*-closed and therefore* $\sigma(A, A_*)_b$*-closed.*

Proof. We show that -* is continuous for the weak operator topology on any von Neumann algebra A on a Hilbert space $\mathcal{H}$. Recall that the the weak operator topology is defined to be the coarsest topology such that for all $\psi, \phi \in \mathcal{H}$ the functional $a \in A \mapsto \langle \psi, a\phi \rangle$ is continuous [69, Definition 5.1.1]. We show that -^* is continuous by showing it preserves limits of nets. Let $(a_i)_{i\in I}$ be a net in A, converging with respect to the weak operator topology to $a \in A$, *i.e.* for all $\psi, \phi \in \mathcal{H}$, we have $\langle \psi, a_i\phi \rangle \to \langle \psi, a\phi \rangle$ in $\mathbb{C}$. Then

$$\langle \psi, a_i\phi \rangle = \langle a_i^*\psi, \phi \rangle = \overline{\langle \phi, a_i^*\psi \rangle}$$

so we have $\overline{\langle \phi, a_i^*\psi \rangle} = \overline{\langle \phi, a^*\psi \rangle}$. As $\bar{\text{-}} : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, we therefore have, for all $\phi, \psi \in \mathcal{H}$:

$$\langle \phi, a_i^*\psi \rangle \to \langle \phi, a^*\psi \rangle,$$

so $a_i^* \to a^*$ in the weak operator topology. As this holds for an arbitrary $a \in A$, we have -^* is continuous in the weak operator topology.

The weak operator topology agrees with the $\sigma(A, A_*)$-topology on the unit ball [69, Theorem 7.4.2], their Smithifications are the same, $\sigma(A, A_*)_b$. Because $\sigma(A, A_*)_b$ is finer than the weak operator topology, $\text{-}^* : A \to \overline{A}$ is continuous from $\sigma(A, A_*)_b$ to the weak operator topology. By Corollary 3.2.16 $\text{-}^* : A \to \overline{A}$ is $\sigma(A, A_*)_b$-continuous for any von Neumann algebra A. We then use the fact that every W*-algebra is linearly homeomorphic to a von Neumann algebra by a normal *-homomorphism [128, Theorem III.3.5]. Using Corollary 3.2.16 again, this is true for the Smithifications of the $\sigma(A, A_*)$ and ultraweak topologies also, so -^* is continuous in this topology for all W*-algebras.

We therefore have $\mathrm{SA}(A)$ is $\sigma(A, A_*)_b$-closed because $\mathrm{SA}(A) = (\text{-}^* - \mathrm{id}_A)^{-1}(\{0\})$, the preimage of a closed set by a continuous function. It is therefore a Smith space by Lemma 3.2.1.

For the positive cone, recall that a bounded operator a on a Hilbert space $\mathcal{H}$ is positive iff for all $\phi \in \mathcal{H}$ we have $\langle \phi, a\phi \rangle \geq 0$ [29, §1.6.7]. Therefore this is true for any element of a von Neumann algebra A on $\mathcal{H}$. We therefore have

$$A_+ = \bigcap_{\phi \in \mathcal{H}} \langle \phi, \text{-}\phi \rangle^{-1}([0, \infty)),$$

which is closed in the weak operator topology because it is an intersection of closed sets. As $\sigma(A, A_*)$ and $\sigma(A, A_*)_b$ are finer than the weak operator topology, A_+ is closed in them as well, for any von Neumann algebra. This is therefore true for all W*-algebras by the same argument seen above. □

Proposition 3.6.2. *For each W*-algebra A,* $(\mathrm{SA}(A), \sigma(A, A_*)_b, A_+, 1_A)$ *is a Smith order-unit space. Defining* $\mathrm{SA}(f) = f|_{\mathrm{SA}(A)}$ *for any normal PU-map or positive subunital map* $f : A \to B$ *defines functors* $\mathbf{W^*Alg}_{\mathrm{PU}} \to \mathbf{SOUS}$ *and* $\mathbf{W^*Alg}_{\mathrm{P}\leq 1} \to \mathbf{SOUS}_{\leq 1}$. *These functors are full and faithful.*

Proof. We have already seen in Proposition 1.2.10 that $(\mathrm{SA}(A), A_+, 1_A)$ is a Banach order-unit space, and Lemma 3.6.1 shows that it is a Smith space with respect to $[-1_A, 1_A]$ and A_+ is closed, so $(\mathrm{SA}, \sigma(A, A_*)_b, A_+, 1_A)$ is a Smith order-unit space.

We have $\mathrm{SA}(f) = f_{\mathrm{SA}(A)}$ defines a functor in each case by combining Proposition 1.2.10 with the fact that the composition of continuous functions is continuous.

Faithfulness follows directly from the faithfulness in Proposition 1.2.10, so we only need to show fullness. Let $g : \mathrm{SA}(A) \to \mathrm{SA}(B)$ be a morphism in **SOUS**. As in Proposition 1.2.10, we define

$$f(a) = g(a_{\Re}) + ig(a_{\Im}),$$

and this is a map in $\mathbf{C^*Alg}_{\mathrm{PU}}(A, B)$, or $\mathbf{C^*Alg}_{\mathrm{P}\leq 1}(A, B)$ such that $\mathrm{SA}(f) = g$. Therefore we only need to show that f is continuous, from $\sigma(A, A_*)$ to $\sigma(B, B_*)$. We first remark that the formulas defining $a_{\Re}$ and $a_{\Im}$ show that the mappings $-_{\Re}$ and $-_{\Im}$ are continuous in $\sigma(A, A_*)_b$, because addition and scalar multiplication are continuous in any topological vector space, and $-^*$ is continuous by Lemma 3.6.1.

Let $(a_j)_{j\in J}$ be a net in A converging to $a \in A$ in $\sigma(A, A_*)_b$. We have

$$f(a_j) = g((a_j)_{\Re}) + ig((a_j)_{\Im})$$

By the previous paragraph, $(a_j)_{\Re} \to a_{\Re}$, so since g is continuous $g((a_j)_{\Re}) \to g(a_{\Re})$, and similarly $g((a_j)_{\Im}) \to g(a_{\Im})$. By continuity of addition and scalar multiplication

$$f(a_j) = g((a_j)_{\Re}) + ig((a_j)_{\Im}) \to g(a_{\Re}) + ig(a_{\Im}) = f(a),$$

and therefore f is continuous from $\sigma(A, A_*)_b$ to $\sigma(B, B_*)_b$. By Corollary 3.2.24, f is continuous from $\sigma(A, A_*)$ to $\sigma(B, B_*)$ so is a map in $\mathbf{W^*Alg}_{\mathrm{PU}}$. □

Corollary 3.6.3. *The functor* $\mathrm{NS} : \mathbf{W^*Alg}_{\mathrm{PU}} \to \mathcal{EM}(\mathcal{D})$ *and the functor* $\mathrm{NS}_{\leq 1} : \mathbf{W^*Alg}_{\mathrm{P}\leq 1} \to \mathcal{EM}(\mathcal{D}^{\leq 1})$ *are fully faithful.*

Proof. The categories of W*-algebras embed fully and faithfully in **SOUS** and $\mathbf{SOUS}_{\leq 1}$ by the functor SA (Proposition 3.6.2). The functor $\mathrm{NS} = B \circ G^{\beta} \circ \mathrm{SA}$ and $\mathrm{NS}_{\leq 1} = B^{\leq 1} \circ G^{\beta} \circ \mathrm{SA}$. Now, G^{β} is an equivalence by Theorem 3.4.5, hence is full and faithful. By Proposition 2.4.8, B and $B^{\leq 1}$ are full and faithful, so the composite functors NS and $\mathrm{NS}_{\leq 1}$ are full and faithful. □

The preceding corollary has been used in [111], where it is used to show that the category of W*-algebras with *completely positive* unital maps embeds contravariantly into the category of quantum predomains defined there.

We will use the following standard result later, and we state it here because it is simplest to give the proof here.

Lemma 3.6.4. *If A is a W*-algebra, and $a, b \in A$, then if $\phi(a) = \phi(b)$ for all $\phi \in \mathrm{NS}(A)$, we have $a = b$. Alternatively, A is separated by normal states.*

Proof. We know from Theorem 3.4.11 and Proposition 3.6.2 that $\epsilon_{\mathrm{SA}(A)} : \mathrm{SA}(A) \to \mathrm{BAff}(\mathrm{CStat}(\mathrm{SA}(A)))$ is an isomorphism. Recall that $\mathrm{CStat}(\mathrm{SA}(A)) = \mathrm{NS}(A)$. We therefore have that if $a \in \mathrm{SA}(A)$, and for all $\phi \in \mathrm{NS}(A)$ $\phi(a) = 0$, then

$\epsilon_{\mathrm{SA}(A)}(a)(\phi) = 0$ for all $\phi \in \mathrm{NS}(A)$ and so $\epsilon_{\mathrm{SA}(A)}(a) = \epsilon_{\mathrm{SA}(A)}(0)$, which by bijectivity of $\epsilon_{\mathrm{SA}(A)}$ implies $a = 0$.

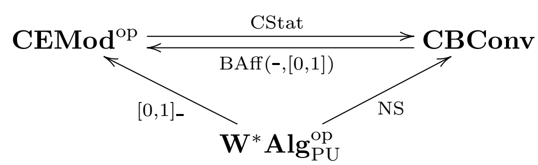
We extend this to $a \in A$ as follows. Suppose $\phi(a) = 0$ for all $\phi \in \mathrm{NS}(A)$. We can decompose $a = a_{\Re} + ia_{\Im}$ with $a_{\Re}, a_{\Im} \in \mathrm{SA}(A)$ by Lemma 1.2.2. Since each ϕ is a state, it preserves self-adjointness (Lemma 1.2.3, so

$$0 = \phi(a_{\Re} + ia_{\Im}) = \phi(a_{\Re}) + i\phi(a_{\Im})$$

implies that $\phi(a_{\Re}) = 0$ and $\phi(a_{\Im}) = 0$. By the previous paragraph, we therefore have $a_{\Re} = a_{\Im} = 0$, and so $a = 0$.

Now let $a, b \in A$ and suppose that $\phi(a) = \phi(b)$ for all $\phi \in \mathrm{NS}(A)$. By linearity, $\phi(a - b) = 0$ for all $\phi \in \mathrm{NS}(A)$, so $a - b = 0$, *i.e.* $a = b$. □

We also get a triangle of adjunctions for the state and predicate transformer semantics of quantum programs interpreted in $\mathbf{W^*Alg}_{\mathrm{PU}}$:



This time we use Theorems 3.4.13 and 3.4.5, as well as the necessary results about W*-algebras above. We should mention that, using an alternative definition of normal map of W*-algebras in order-theoretic terms, Mathys Rennela produced a state and effect triangle for $\mathbf{W^*Alg}_{\mathrm{P}\leq 1}$ in [110, Theorem 4.1]. However, the top line is only an adjunction, rather than an equivalence. We do not know whether the order-theoretic definition of normal map can be used to produce an equivalence in the general case, as the proof for W*-algebras uses the representation as operators on a Hilbert space, which is why we use the approach via Smith order-unit spaces.

Chapter 4

Compact Convex Sets, $\mathcal{R}$ and $\mathcal{E}$

This chapter originated in the paper "The Expectation Monad in Quantum Foundations"[64] by Bart Jacobs, Jorik Mandemaker and the author, as well as its original version [63] with only Jacobs and Mandemaker. The part on the Radon monad originates in [45]. The part on compact effect modules is original.

4.1 Introduction

In the last chapter we saw that sequentially complete bounded convex subsets of locally convex spaces, or equivalently the bases of Banach base norm spaces, can be embedded as a reflective subcategory of $\mathcal{EM}(\mathcal{D})$ and $\mathcal{EM}(\mathcal{D}_\infty)$. We gave no corresponding result for compact convex subsets, or equivalently bases of Smith base-norm spaces. We have also seen that $\mathcal{EM}(\mathcal{R})$ and $\mathcal{EM}(\mathcal{E})$ can both be seen as categories of compact convex sets, and so should be related to **CCL** in some way. In this chapter, we use some results due to Świrszcz to prove that **CCL** $\simeq \mathcal{EM}(\mathcal{R}) \simeq \mathcal{EM}(\mathcal{E})$. This justifies the notion that these are a legitimate notion of convex set, as they occur in many different guises, and also that the embedding in a locally convex space can be considered as merely a property rather than a structure.

We can then apply these results to reformulate **CEMod** in two different ways in an embedding-independent fashion.

4.2 Świrszcz's Theorem for $\mathcal{R}$

In this section we show that the Radon monad arises from an adjunction in [126] enabling us to use Świrszcz's theorem 3 from that paper to show that the categories **CCL** and $\mathcal{EM}(\mathcal{R})$ are equivalent, which we can then apply to represent noncommutative C*-algebras. The adjunction in question has

$U\colon \mathbf{CCL} \to \mathbf{CHaus}$ as the right adjoint, and the details of the construction of the left adjoint are not given. In order to prove that $\mathcal{R}$ is the monad arising from this adjunction, we need to know its unit and counit, so our next task is to define the left adjoint explicitly. Of course, any other left adjoint will be naturally isomorphic (Proposition 0.4.2).

We begin as follows. We define $\acute{\mathcal{S}}\colon \mathbf{CHaus} \to \mathbf{CCL}$ as $\acute{\mathcal{S}} = \mathrm{Stat} \circ C$. Hence $\mathcal{R} = U \circ \acute{\mathcal{S}}$. To show that $\acute{\mathcal{S}}$ is the left adjoint to U, we use the unit and counit definition of an adjunction (Theorem 0.4.1 (iv)). We already know the unit, $\eta_X : X \to U(\acute{\mathcal{S}}(X))$, as we gave it when defining the unit of $\mathcal{R}$. To define the counit we use the notion of *barycentre*.

We can understand the intuitive notion of barycentre by thinking of a (Radon) probability measure μ on the unit square $[0,1]^2$. If we wanted to find the centre of mass of μ, which we shall call $b \in [0,1]^2$, we would take

$$b_x \quad = \int\limits_{[0,1]^2} x\mathrm{d}\mu \qquad \text{and} \quad b_y \quad = \int\limits_{[0,1]^2} y\mathrm{d}\mu$$

for the x and y coordinates. We can see that x and y are continuous affine functions from $[0,1]^2 \to \mathbb{R}$, assigning each point to its x and y coordinate respectively. Therefore we can rewrite the above as

$$\int\limits_{[0,1]^2} x\mathrm{d}\mu \quad = \quad x(b) \qquad \text{and} \quad \int\limits_{[0,1]^2} y\mathrm{d}\mu \quad = \quad y(b).$$

In monadic terms, this means that both projections $\pi_1, \pi_2\colon [0,1]^2 \to [0,1]$ are maps of Eilenberg-Moore algebras for the Radon monad, in the sense that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{R}([0,1]^2) & \xrightarrow{\mathcal{R}(\pi_i)} & \mathcal{R}([0,1]) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ [0,1]^2 & \xrightarrow[\pi_i]{} & [0,1] \end{array}$$

We write α for the algebra $\nu \mapsto \int \mathrm{id}\mathrm{d}\nu$, see also [58], and β for the product algebra structure, given by $\mu \mapsto \langle \int \pi_1 \mathrm{d}\mu, \int \pi_2 \mathrm{d}\mu \rangle = \langle \int x\mathrm{d}\mu, \int y\mathrm{d}\mu \rangle$.

In fact, any affine continuous $\mathbb{R}$-valued function a on $[0,1]^2$ can be expressed as $\alpha x + \beta y + \gamma$ for $\alpha, \beta, \gamma \in \mathbb{R}$, and so

$$\begin{aligned} \int_{[0,1]^2} (\alpha x + \beta y + \gamma) d\mu &= \alpha \int_{[0,1]^2} x d\mu + \beta \int_{[0,1]^2} y d\mu + \gamma \int_{[0,1]^2} d\mu \\ &= \alpha x(b) + \beta y(b) + \gamma \\ &= (\alpha x + \beta y + \gamma)(b), \end{aligned}$$

or $\int_{[0,1]^2} a d\mu = a(b)$ for all affine continuous functions $a : [0,1]^2 \to \mathbb{R}$. If we use the linear functional definition of a Radon measure, we have motivated the following definition.

Definition 4.2.1. *If $X \in \mathbf{CCL}$ and $\phi \in \acute{\mathcal{S}}(U(X))$, then a point $x \in X$ is a* barycentre *for ϕ if for all continuous affine functions a from $X \to \mathbb{R}$ we have that $\phi(a) = a(x)$.* □

To handle barycentres, and for some other purposes, will require the following important lemma, one of several variants of the Hahn-Banach separation lemma, and some of its corollaries, which give an affine analogue of Urysohn's lemma for objects in **CCL**.

Lemma 4.2.2. *If E is a locally convex topological vector space, X a closed convex subset and Y a compact convex subset that is disjoint from X, then there exists a continuous linear functional $\phi\colon E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.* □

For a proof, see either [24, theorem IV.3.9] or [118, II.4.2 corollary 1].

Corollary 4.2.3. *Let $(E, K) \in \mathbf{CCL}$. In the following X, Y will be arbitrary closed disjoint convex subsets of K, x, y arbitrary distinct points of K.*

(i) There is a $\phi \in \mathrm{CAff}(K)$ and an $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.

(ii) There is a $\phi \in \mathrm{CAff}(K)$ such that $\phi(x) \neq \phi(y)$.

(iii) There is a $\phi \in \mathbf{CCL}(K, [0, 1])$ and an $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, 1]$ and $\phi(Y) \subseteq [0, \alpha)$.

(iv) There is a $\phi \in \mathbf{CCL}(K, [0, 1])$ such that $\phi(x) \neq \phi(y)$.

Proof.

(i) Apply Lemma 4.2.2 to obtain $\phi'\colon V \to \mathbb{R}$ separating X from Y. Since K has the subspace topology, $\phi = \phi'|_K$ is continuous, and since ϕ' is linear, ϕ is affine, hence $\phi \in \mathrm{CAff}(K)$. We also keep the properties that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.

(ii) This follows directly from (i), using the fact that points are compact and convex.

(iii) We use (i) and obtain $\phi' \in \mathrm{CAff}(K)$ and $\alpha' \in \mathbb{R}$. Since the image of a compact space is compact, and a compact subset of $\mathbb{R}$ is closed and bounded, the numbers

$$\beta_\uparrow = \sup \phi'(K) \qquad\qquad \beta_\downarrow = \inf \phi'(K)$$

exist, and ϕ' can be considered as an affine continuous map $K \to [\beta_\downarrow, \beta_\uparrow]$. We define

$$\phi(k) = \frac{\phi(k) - \beta_\downarrow}{\beta_\uparrow - \beta_\downarrow}$$

if $\beta_\uparrow \neq \beta_\downarrow$, otherwise we define it without dividing by anything, though this can only happen if one of X or Y is empty. The image of ϕ is contained in $[0,1]$, and ϕ is affine and continuous, being the composition of affine and continuous maps. We define

$$\alpha = \frac{\alpha' - \beta_\downarrow}{\beta_\uparrow - \beta_\downarrow}$$

again not doing the division if it is zero. We have that $\phi(X) \subseteq (\alpha, \infty)$, and since the image of ϕ is contained in $[0,1]$, this implies $\phi(X) \subseteq (\alpha, 1]$. The proof that $\phi(Y) \subseteq [0, \alpha)$ is similar.

(iv) This is proven using (iii), again using the fact that points are closed, convex sets. □

Using the properties proven above, we can start to define the counit of the adjunction. For $(E,X) \in \mathbf{CCL}$, we define $\epsilon_X : \acute{\mathcal{S}}(U(X)) \to X$ to map a Radon measure ϕ to a barycentre in X. As yet, we did not show that a barycentre exists for every measure or that it is unique if it exists. We address the second point first.

Lemma 4.2.4. *For every $\phi \in \acute{\mathcal{S}}(U(X))$ the barycentre is unique,* i.e. *any two barycentres of ϕ are equal. Therefore $\varepsilon_X \colon \acute{\mathcal{S}}(U(X)) \to X$ mapping ϕ to its barycentre is at least a partial function.*

Proof. Let (E,X) be an object of $\mathbf{CCL}$, E being the locally convex space and X the compact convex subset. Suppose for a contradiction that $x, x' \in X$ are barycentres of $\phi \in \acute{\mathcal{S}}(U)$, and $x \neq x'$. By Corollary 4.2.3 (ii), there is an $f \in \mathrm{CAff}(X)$ such that $f(x) \neq f(x')$. Since x and x' are both barycentres of ϕ,

$$f(x) = \phi(f) = f(x')$$

a contradiction. □

In fact, it is well known that Radon measures on compact convex subsets of locally convex spaces always have barycentres [4, Proposition I.2.1 and I.2.2]. In [45] we used this theorem and then proved that the mapping ϵ_X was continuous and affine in a separate step. Using the results of Chapter 3, we can in fact show this in one step. This result is also shown in [102, Proposition 1.1] and [37, Proposition 7.1], but we include it here for the convenience of the reader.

In the following we use i_X to refer to the inclusion mapping $\mathrm{CAff}(X) \hookrightarrow C(U(X))$, where $(E,X) \in \mathbf{CCL}$, which is a positive unital map by definition.

Lemma 4.2.5. *For each $(E,X) \in \mathbf{CCL}$, every Radon measure $\phi \in \acute{\mathcal{S}}(U(X))$ has a barycentre in X. The mapping $\varepsilon_X : \acute{\mathcal{S}}(U(X)) \to X$ is an affine, continuous map,* i.e. *a map in $\mathbf{CCL}$.*

Proof. We first show that this is true for bases of Smith base-norm spaces, which by Proposition 3.3.3 gives us *essentially* all objects of $\mathbf{CCL}$. We show that this is true by showing that for any Smith base-norm space $(E, \mathcal{T}, E_+, \tau)$ the composite map

$$\acute{\mathcal{S}}(B(E)) \xrightarrow{\mathrm{Stat}(i)} \mathrm{Stat}(\mathrm{CAff}(B(E))) \xrightarrow{\mathrm{Stat}(\rho_E)} \mathrm{Stat}(F^\beta(E)) \xrightarrow{B(\eta_E^{-1})} B(E),$$

where ρ_E is the restriction isomorphism from Proposition 3.3.5, and η_E the unit of the adjoint equivalence from Theorem 3.3.7, maps a Radon measure ϕ to a barycentre of it in $B(E)$. It follows from this that $\varepsilon_{B(E)}$ is a total function and is a morphism in **CCL**.

We need to show that for all $\phi \in \acute{\mathcal{S}}(B(E))$, and all $a \in \mathrm{CAff}(B(E))$

$$\phi(a) = a(B(\eta_E^{-1}) \circ \mathrm{Stat}(i \circ \rho_E)(\phi)).$$

We start with the right hand side:

$$\begin{aligned}
a(B(\eta_E^{-1})(\mathrm{Stat}(i \circ \rho_E)(\phi)) &= a(B(\eta_E^{-1})(\phi \circ i \circ \rho_E)) \\
&= a(\eta_E^{-1}(\phi \circ i \circ \rho_E)) \\
&= a(\eta_E^{-1}(\phi \circ \rho_E)) && i \text{ an inclusion} \\
&= \eta_E(\eta_E^{-1}(\phi \circ \rho_E))(\rho_E^{-1}(a)) && \text{definition of } \eta_E \text{ and } \rho_E \\
&= (\phi \circ \rho_E)(\rho_E^{-1}(a)) \\
&= \phi(a).
\end{aligned}$$

We now extend this to all objects in **CCL**. We use Proposition 3.3.3, specifically the fact that every object $(E, X) \in$ **CCL** is isomorphic to $(F, B(F))$ for some Smith base-norm space F. To avoid cumbersome notation, we prove instead that if (F, Y) is an object in **CCL** such that $\varepsilon_Y : \acute{\mathcal{S}}(Y) \to Y$ is total and a map in **CCL**, and $(E, X) \in$ **CCL** is equipped with a **CCL**-isomorphism $f : X \to Y$, then

$$\acute{\mathcal{S}}(X) \xrightarrow{\acute{\mathcal{S}}(f)} \acute{\mathcal{S}}(Y) \xrightarrow{\varepsilon_Y} Y \xrightarrow{f^{-1}} X$$

maps a Radon measure $\phi \in \acute{\mathcal{S}}(X)$ to its barycentre in X, and therefore ε_X is total and a morphism in **CCL**.

Therefore we want to show that for all $\phi \in \acute{\mathcal{S}}(X)$ and $a \in \mathrm{CAff}(X)$ that $\phi(a) = a(f^{-1}(\varepsilon_Y(\acute{\mathcal{S}}(f)(\phi))))$. As $a \circ f^{-1}$ is the composite of affine continuous functions, it is affine and continuous and therefore an element of $\mathrm{CAff}(Y)$. By the definition of barycentre, we have

$$(a \circ f^{-1})(\varepsilon_Y(\acute{\mathcal{S}}(f)(\phi))) = \acute{\mathcal{S}}(f)(\phi)(a \circ f^{-1}) = \phi(a \circ f^{-1} \circ f) = \phi(a).$$

□

Lemma 4.2.6. *The family $\{\varepsilon_X\}$ defines a natural transformation $\varepsilon : \acute{\mathcal{S}} \circ U \Rightarrow$ Id.*

Proof. We must show that

$$\begin{array}{ccc} \acute{\mathcal{S}}(U(X)) & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle \acute{\mathcal{S}}(U(f))}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \acute{\mathcal{S}}(U(Y)) & \xrightarrow[\epsilon_Y]{} & Y \end{array}$$

Suppose that $\phi \in \acute{\mathcal{S}}(U(X))$ and $\varepsilon_X(\phi) = x$, i.e. x is the barycentre of ϕ. It suffices to show that $f(x)$ is the barycentre of $\acute{\mathcal{S}}(U(f)(\phi)$. Let $h \in C(Y)$, and we have by definition that

$$\acute{\mathcal{S}}(U(f))(\phi)(h) = \phi(h \circ f)$$

We want to show that if h is affine, then $\acute{\mathcal{S}}(U(f))(\phi)(h) = h(f(x))$, as this would show $f(x)$ is the barycentre. Since $h \circ f$ is the composite of continuous, affine functions, it is also continuous and affine, and so, using the fact that x is the barycentre of ϕ, we have that $\phi(h \circ f) = (h \circ f)(x) = h(f(x))$, which is what we were required to prove. ☐

Taken together, the preceding three lemmas define the counit. We can now move on to showing that this is actually an adjunction.

Theorem 4.2.7. *The functor* $\acute{\mathcal{S}}$: **CHaus** $\to$ **CCL** *is the left adjoint to* U : **CCL** $\to$ **CHaus**

Proof. We show that the unit-counit diagrams commute (Theorem 0.4.1 (v)).

The first diagram is:

$$\begin{array}{ccc} UY & \xrightarrow{\eta_{UY}} & U(\acute{\mathcal{S}}(U(Y))) \\ & {\scriptstyle \mathrm{id}_{UY}}\searrow & \big\downarrow{\scriptstyle U\varepsilon_Y} \\ & & UY \end{array}$$

To show that it commutes, we must show that for all $y \in UY$, y is the barycentre of $\eta_{UY}(y)$. Using the definition of η, we have that for any affine continuous function $a : X \to \mathbb{R}$ that

$$\eta_{UY}(x)(a) = a(x)$$

because that is already true for any continuous functions $a \in C(X)$. Therefore x is the barycentre of $\eta_{UY}(x)$, and so the diagram commutes.

The second diagram we must consider is the following:

$$\begin{array}{ccc} \acute{\mathcal{S}}(X) & \xrightarrow{\acute{\mathcal{S}}(\eta_X)} & \acute{\mathcal{S}}(U(\acute{\mathcal{S}}(X))) \\ & {\scriptstyle \mathrm{id}_{\acute{\mathcal{S}}(X)}}\searrow & \big\downarrow{\scriptstyle \epsilon_{\acute{\mathcal{S}}(X)}} \\ & & \acute{\mathcal{S}}(X) \end{array}$$

This time, we need to show that $\phi \in \acute{\mathcal{S}}(X)$ is the barycentre of $\acute{\mathcal{S}}(\eta_X)(\phi)$. So consider an affine continuous function $a : \acute{\mathcal{S}}(X) \to \mathbb{R}$. We want to show that $\acute{\mathcal{S}}(\eta_X)(\phi)(a) = a(\phi)$ for all $\phi \in \acute{\mathcal{S}}(X)$. To do this, we use Lemma 1.5.5. We show the diagram commutes on the convex combinations of extreme points, and since this is a dense subset, the diagram commutes by continuity. So let $\{x_1, \dots x_n\}$ be a finite subset of X, and

$$\sum_{i=1}^{n} \alpha_i \eta_X(x_i)$$

a finite convex combination of extreme points of $\acute{\mathcal{S}}(X)$. Now

$$\begin{aligned}
\acute{\mathcal{S}}(\eta_X)\left(\sum_{i=1}^{n} \alpha_i \eta_X(x_i)\right)(a) &= \left(\sum_{i=1}^{n} \alpha_i \eta_X(x_i)\right)(a \circ \eta_X) \\
&= \sum_{i=1}^{n} \alpha_i \eta_X(x_i)(a \circ \eta_X) \\
&= \sum_{i=1}^{n} \alpha_i a(\eta_X(x_i)) \\
&= a\left(\sum_{i=1}^{n} (\eta_X(x_i))\right)
\end{aligned}$$

with the last step holding because a is an affine function.

As explained before, this shows $\acute{\mathcal{S}}(\eta_X)(\phi)(a) = a(\phi)$ for all $\phi \in \acute{\mathcal{S}}(X)$, and hence the diagram commutes. Thus we have that $\acute{\mathcal{S}}$ is the left adjoint to U. □

Now that we have defined the adjunction $\acute{\mathcal{S}} \dashv U$, we can move on to proving that $\mathcal{R}$ is not only the same functor as the monad derived from $\acute{\mathcal{S}} \dashv U$ but also the same as a monad. In order to do this, we require a few lemmas concerning the definition of μ we gave at the start of Section 1.5. The map μ was defined using the map $\psi \mapsto \psi(a)$. In fact, this map is simply $\epsilon_{\mathrm{SA}(A)}$, using the counit from Theorem 3.3.8.

When defining μ_X for the Radon monad, we were using $\epsilon_{C_{\mathbb{R}}(X)}$ for a compact Hausdorff space X, since $\mathrm{SA}(C(X)) = C_{\mathbb{R}}(X)$, the real-valued functions. We can see that

$$\mu_X(\Phi)(a) = \Phi(\epsilon_{C_{\mathbb{R}}(X)}(a)). \tag{4.1}$$

Theorem 4.2.8. *The monad* : **CHaus** → **CHaus** *given by* $\acute{\mathcal{S}} \dashv U$ *is the Radon monad* $\mathcal{R}$.

Proof. We have by definition that $\mathcal{R} = U\acute{\mathcal{S}}$ and $\eta = \eta$. Therefore we only need to show that $\mu = U\varepsilon\acute{\mathcal{S}}$. What we need to show, then, is that if X is a compact Hausdorff space and $\Phi \in \acute{\mathcal{S}}(U(\acute{\mathcal{S}}(X)))$, then $\mu(\Phi)$ is the barycentre of Φ. That is to say, for all $a \in \mathrm{CAff}(\acute{\mathcal{S}}(X))$, $\Phi(a) = a(\mu_X(\Phi))$. By Theorem 3.3.8, every $a \in \mathrm{CAff}(\acute{\mathcal{S}}(X))$ is of the form $\epsilon_{C_{\mathbb{R}}(X)}(b)$ for some $b \in C_{\mathbb{R}}(X)$. Substituting this expression for a, we want to show that $\Phi(\epsilon_{C_{\mathbb{R}}(X)}(b)) = \epsilon_{C_{\mathbb{R}}(X)}(b)(\mu_X(\Phi))$. Starting with the right hand side and using (4.1) we get

$$\epsilon_{C_{\mathbb{R}}(X)}(b)(\mu_X(\Phi)) = \mu_X(\Phi)(b) = \Phi(\epsilon_{C_{\mathbb{R}}(X)}(b))$$

as required. □

Theorem 4.2.9 (Świrszcz's theorem)**.** *The forgetful functor* $U\colon \mathbf{CCL} \to \mathbf{CHaus}$ *is monadic,* i.e. $\mathbf{CCL} \simeq \mathcal{EM}(U \circ \acute{\mathcal{S}})$. *By Theorem 4.2.8,* $\mathbf{CCL} \simeq \mathcal{EM}(\mathcal{R})$. □

This comes from [126, Theorem 3]. A proof not using any monadicity theorems can be found in [120, Proposition 7.3], and another proof may be found in [74, Theorem 8.5] where the subprobabilistic case is also treated.

At this point the reader might wonder what rôle the embedding in a locally convex space plays. If X is a compact convex subset of a topological vector space, $\mathcal{R}(X)$ is still defined, and we have a map $\mathcal{D}(X) \to X$ by the convexity, so we have a partially defined map on the dense subset $\tau_X(\mathcal{D}(X))$ (Lemma 1.5.5) of $\mathcal{R}(X)$ to X. If it were always possible to extend this to all of $\mathcal{R}(X)$, for instance by proving it was uniformly continuous, then every compact convex subset of a topological vector space would be an $\mathcal{R}$-algebra, and therefore be embeddable in a locally convex space. However, there are compact convex subsets of non-locally compact spaces that have no extreme points. If they were embeddable in a locally convex space this would contradict the Krein-Milman theorem [24, Proposition 7.4]. The first example was given by Roberts [112], and later he constructed an example in L^p for $0 < p < 1$, a metrizable topological vector space that is not locally convex. A more recent example is given in [72, Theorem 4.3]. Such convex sets are necessarily not "observable" in the sense of [64], *i.e.* there are pairs of points that cannot be distinguished by any continuous affine map to $[0, 1]$ (or equivalently $\mathbb{R}$).

4.3 Świrszcz's Theorem for $\mathcal{E}$

We introduce at this point the functor $\mathbf{Set} \to \mathbf{BOUS}^{\mathrm{op}}$ analogous to C : $\mathbf{CHaus} \to \mathbf{BOUS}^{\mathrm{op}}$. We write $\ell^\infty(X)$ for the set of functions $\phi\colon X \to \mathbb{R}$ which are bounded: there is an $N \in \mathbb{N}$ with $|\phi(x)| \leq N$ for all $x \in X$. These functions form an ordered vector space, via pointwise operations and order. The function $u\colon X \to \mathbb{R}$ with $u(x) = 1$ is a strong unit that is Archimedean. The induced norm is the *uniform* or *supremum* norm $\|\phi\|_\infty = \sup\{|\phi(x)| \mid x \in X\}$. It is not hard to see that $\ell^\infty(X)$ is complete in this norm, and thus a Banach order-unit space.

For the category **CHaus** we have seen a monadicity result over **Set** in Subsection 0.4.1, and for **CCL** we have seen a monadicity result over **CHaus** (Theorem 4.2.9). There in fact a monadicity result for **CCL** over **Set**, also due to Świrszcz.

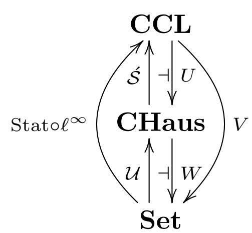
Theorem 4.3.1 (Świrszcz)**.** *The category* **CCL** *of compact convex sets is monadic over* **Set***, where the left adjoint to the forgetful functor* **CCL** $\to$ **Set** *is the following composite:*

$$\mathcal{S} = \big(\mathbf{Set} \xrightarrow{\mathcal{U}} \mathbf{CHaus} \xrightarrow{\acute{\mathcal{S}}} \mathbf{CCL}\big).$$

obtained by composing the adjunctions from Subsection 0.4.1 and Theorem 4.2.7. □

The proof in [126] uses Linton's monadicity theorem. A more elementary proof (of monadicity over **CHaus**, but adaptable to this case) can be found in [120].

However, we cannot use this result immediately because the monad arising from the adjunction above is not $\mathcal{E}$. To help with this, we will show that $\mathrm{Stat} \circ \ell^\infty \dashv V$ in the diagram below:



The monad $V\mathrm{Stat}\ell^\infty$ will be more easily related to $\mathcal{E}$. We define the unit of the adjunction first.

Lemma 4.3.2. *The map* $\eta_X : X \to V(\mathrm{Stat}(\ell^\infty(X)))$*, defined as follows*

$$\eta_X(x)(a) = a(x),$$

where $x \in X$ *and* $a \in \ell^\infty(X)$*, is a natural transformation* $\eta : \mathrm{Id} \Rightarrow V\mathrm{Stat}\ell^\infty$*.*

Proof. First we must check that $\eta_X(x)$ is a state. It is linear by the pointwiseness of the definitions of addition and scalar multiplication. We can show it is positive because the positive elements of $\ell^\infty(X)$ are just the non-negative functions, and the unit is 1, so $\eta_X(x)$ preserves positive elements and the unit. Therefore η_X defines a function in **Set**, so we move on to verifying that it is natural. Let $f : X \to Y$ be a function, and we want to show the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & U\mathrm{Stat}\ell^\infty(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle U\mathrm{Stat}\ell^\infty(f)} \\ Y & \xrightarrow[\eta_Y]{} & U\mathrm{Stat}\ell^\infty(Y). \end{array}$$

To do so, let $x \in X$ and $b \in \ell^\infty(Y)$. For the lower left route we have

$$(\eta_Y \circ f)(x)(b) = \eta_Y(f(x))(b) = b(f(x)),$$

while for the upper right route we have

$$\begin{aligned}(U\mathrm{Stat}\ell^\infty(f)\circ\eta_X)(x)(b) &= U(\mathrm{Stat}(\ell^\infty(f)))(\eta_X(x))(b) = \eta_X(x)(\ell^\infty(f)(b)) \\ &= \ell^\infty(f)(b)(x) = b(f(x)),\end{aligned}$$

hence the diagram commutes. □

We can extend the notion of barycentre as follows. For $(E, X) \in \mathbf{CCL}$ and $\phi \in \mathrm{Stat}(\ell^\infty(V(X)))$ we say $x \in X$ is a *barycentre* of ϕ if for all $a \in \mathrm{CAff}(X)$ we have $\phi(a) = a(x)$. Note that this definition is well formed because $\mathrm{CAff}(X) \subseteq C(X) \subseteq \ell^\infty(X)$ (Proposition A.2.3). By a similar argument to that used in Lemma 4.2.4, the Hahn-Banach separation theorem implies that barycentres are unique if they exist.

Proposition 4.3.3. *The functor* $\mathrm{Stat}\circ\ell^\infty$ *is left adjoint to* $V : \mathbf{CCL} \to \mathbf{Set}$.

Proof. We prove this by defining the counit and verifying the triangle axioms for the unit and counit (Theorem 0.4.1 (v)). We use, from Proposition A.2.3, the inclusion mapping $\iota : C \Rightarrow \ell^\infty W$ (where for the moment $W : \mathbf{CHaus} \to \mathbf{Set}$ is the forgetful functor) to define

$$\epsilon = \varepsilon \circ \mathrm{Stat}\iota,$$

where ε is the counit from Theorem 4.2.7. By definition this is a natural transformation of the correct type. If $\phi \in \mathrm{Stat}(\ell^\infty(V(X)))$ and $a \in \mathrm{CAff}(X)$, we can observe that

$$\begin{aligned}\phi(a) &= \phi(\iota(a)) \\ &= \mathrm{Stat}(\iota)(\phi)(a) \\ &= a(\varepsilon_X(\mathrm{Stat}(\iota)(\phi))) && \text{definition of } \varepsilon \\ &= a(\epsilon_X(\phi)),\end{aligned}$$

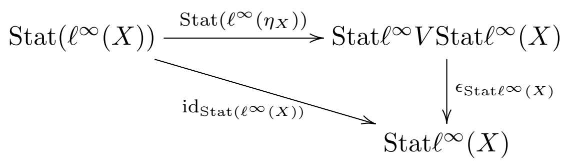
so ϵ_X maps states to their barycentres, where barycentre is taken with the more general sense.

Thus we have defined a unit (in Lemma 4.3.2) and a counit and need to show that they satisfy the unit-counit laws. The first diagram is the following (for $X \in \mathbf{CCL}$)

$$\begin{array}{ccc} VX & \xrightarrow{\eta_{VX}} & V\mathrm{Stat}\ell^\infty VX \\ & {\scriptstyle \mathrm{id}_{VX}} \searrow & \downarrow {\scriptstyle V\epsilon_X} \\ & & VX \end{array}$$

This states that the barycentre of a Dirac measure at x is x. So we must show that for all $x \in X$, and for all $a \in \mathrm{CAff}(X)$, $\eta_{VX}(x)(a) = a(x)$. This follows directly from the definition.

The second diagram is (for X a set):



Let $\phi \in \mathrm{Stat}(\ell^\infty(X))$. To show this diagram commutes, we will show that ϕ is the barycentre of $\mathrm{Stat}(\ell^\infty(\eta_X))(\phi)$. So let $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(X)))$, and we want to show that $\mathrm{Stat}(\ell^\infty(\eta_X))(\phi)(a) = a(\phi)$. By Theorem 3.3.8, $a = \epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)$ for some $b \in \ell^\infty(X)$. We then have that

$$\mathrm{Stat}(\ell^\infty(\eta_X))(\phi)(\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)) = \phi(\ell^\infty(\eta_X)(\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b))) = \phi(\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b) \circ \eta_X).$$

By using $x \in X$, we observe

$$(\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b) \circ \eta_X)(x) = \epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)(\eta_X(x)) = \eta_X(x)(b) = b(x),$$

so $epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b) \circ \eta_X = b$, and we have

$$\mathrm{Stat}(\ell^\infty(\eta_X))(\phi)(\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)) = \phi(b) = \epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)(\phi) = a(\phi).$$

Therefore we have an adjunction. □

Since we have just introduced the barycentre, we take this moment to give an equivalent characterization of it.

Lemma 4.3.4. *Let $X \in \mathbf{CCL}$ and $\phi \in \mathrm{Stat}(\ell^\infty(V(X)))$. A point $x \in X$ is the barycentre of ϕ iff for all $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(V(X))), [0,1])$ $a(x) = \phi(a)$. For convenience, we call this property that of being a* truncated barycentre.

Proof. Because $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(V(X))), [0,1]) \subseteq \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(V(X))))$, if x is the barycentre of ϕ, then $\phi(a) = a(x)$ for all $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(V(X))), [0,1])$. We reduce to proving the converse. Suppose that x is a truncated barycentre of ϕ, and let y be the barycentre. For all $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(V(X))), [0,1])$, we therefore have

$$a(x) = \phi(a) = a(y)$$

By the contrapositive of Corollary 4.2.3 (iv), $x = y$, so x is the barycentre of ϕ. □

We now have, by Lemma 0.4.8, that $V\mathrm{Stat}\ell^\infty \cong V\acute{\mathcal{S}}\mathcal{U}$ as monads, so by Theorem 4.3.1 $\mathcal{EM}(V\mathrm{Stat}\ell^\infty) \cong \mathcal{EM}(V\acute{\mathcal{S}}\mathcal{U}) \simeq \mathbf{CCL}$. The final step is to produce a monad isomorphism $\mathcal{E} \cong V\mathrm{Stat}\ell^\infty$. To do this, we first give an equivalent definition of $\mu^{\mathcal{E}}$.

We will be using the isomorphism of hom sets $\theta_X : \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{R}})$ that is implied to exist by Theorem 1.2.9, so we have given it a name.

Lemma 4.3.5. *For $\Phi \in \mathcal{E}^2(X)$ and $a \in [0,1]^X$, we have*

$$\mu^{\mathcal{E}}(\Phi)(a) = \Phi(W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(a)) \circ \theta_X^{-1})$$

Proof. The original definition (1.4) is

$$\mu^{\mathcal{E}}(\Phi)(a) = \Phi(\phi \in \mathcal{E}(X) \mapsto \phi(a)),$$

and so it suffices to show that

$$\phi \in \mathcal{E}(X) \mapsto \phi(a) = W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(a)) \circ \theta_X^{-1}$$

We start with the right hand side, evaluating it at an arbitrary $\phi \in \mathcal{E}(X)$:

$$\begin{aligned}(W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(a)) \circ \theta_X^{-1})(\phi) &= V([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(a))(\theta_X^{-1}(\phi)) = \epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(a)(\theta_X^{-1}(\phi)) \\ &= \theta_X^{-1}(\phi)(a) = \phi(a).\end{aligned}$$

The last step is true because $\theta_X^{-1}(\phi)$ is only being evaluted at elements of $[0,1]^X$, where it agrees with ϕ. □

For ease of notation, in the following we use (T, η^T, μ^T) for the monad $V\mathrm{Stat}\ell^\infty$.

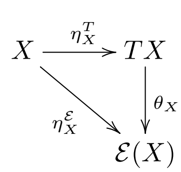
Theorem 4.3.6. *θ is a monad isomorphism $T \Rightarrow \mathcal{E}$. Therefore $\mathcal{EM}(\mathcal{E}) \simeq$ **CCL**.*

Proof. We first observe that the composite functor $V \circ \mathrm{Stat}$ is equal to the hom functor $\mathbf{BOUS}(\text{-}, \mathbb{R})$, and hence $V\mathrm{Stat}\ell^\infty = \mathbf{BOUS}(\text{-}, \mathbb{R}) \circ \ell^\infty$. This is simply because the set is exactly the same set of maps to $\mathbb{R}$ in each case and the action on maps is precomposition in both cases. Now we can use θ_X as follows:

$$\begin{aligned}V\mathrm{Stat}\ell^\infty(X) &= \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{R}}) \\ &= \mathbf{BEMod}([0,1]^X, [0,1]) = \mathcal{E}(X)\end{aligned}$$

and this is in fact a natural isomorphism because the only actual isomorphism is $\theta_X : \mathbf{BOUS}(\ell^\infty(X), \mathbb{R}) \cong \mathbf{BEMod}([0,1]_{\ell^\infty(X)}, [0.1]_{\mathbb{R}})$ which was already proven to be natural, in the other cases the functors agree on both maps and objects and so are trivially naturally isomorphic.

To show it is a monad isomorphism, we must show that the relevant diagrams involving the unit and multiplication commute. First, we must show:



Let $x \in X$ and $a \in [0,1]^X$. Then

$$\theta_X(\eta^T(x))(\phi) = \eta^T(x)(\phi) = \phi(x) = \eta^{\mathcal{E}}(x)(\phi)$$

and so the diagram commutes. We therefore move on to showing that the following diagram commutes:

$$\begin{array}{ccc}
T^2(X) & \xrightarrow{\theta_{TX}} & \mathcal{E}(T(X)) \\
\downarrow{\scriptstyle \mu_X^T} & & \downarrow{\scriptstyle \mathcal{E}(\theta_X)} \\
T(X) & & \mathcal{E}^2(X) \\
& \searrow{\scriptstyle \theta_X} & \downarrow{\scriptstyle \mu_X^{\mathcal{E}}} \\
& & \mathcal{E}(X)
\end{array}$$

The commutativity of the diagram is equivalent to the equation $\mu_X^T = \theta_X^{-1} \circ \mu_X^{\mathcal{E}} \circ \mathcal{E}(\theta_X) \circ \theta_{TX}$, and since $\mu_X^T = V\epsilon_{\mathrm{Stat}(\ell^\infty(X))}$, this will follow if $\theta_X^{-1}(\mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))$ is the barycentre of Φ for each $\Phi \in T^2(X)$. To simplify matters we will use the truncated barycentre characterization from Lemma 4.3.4. Let $a \in \mathrm{CAff}(\mathrm{Stat}(\ell^\infty(X)), [0,1])$ and define $b \in [0,1]^X$ to be such that $[0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(b) = a$ (using Theorem 3.3.8). We have

$$\begin{aligned}
&a(\theta_X^{-1}(\mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))) = \epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}(b)(\theta_X^{-1}(\mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))) \\
&\ = \theta_X^{-1}(\mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))(b) = \mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi)))(b).
\end{aligned}$$

We then use Lemma 4.3.5:

$$\begin{aligned}
&= \mathcal{E}(\theta_X)(\theta_{TX}(\Phi))(W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(b)) \circ \theta_X^{-1}) \\
&= \theta_{TX}(\Phi)(W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(b)) \circ \theta_X^{-1} \circ \theta_X) = \theta_{TX}(\Phi)(W([0,1]_{\epsilon^{\mathbf{BOUS}}_{\ell^\infty(X)}}(b))) \\
&= \theta_{TX}(\Phi)(a) = \Phi(a).
\end{aligned}$$

The last step works because θ is just truncation. We can therefore deduce that $\theta_X^{-1}(\mu_X^{\mathcal{E}}(\mathcal{E}(\theta_X)(\theta_{TX}(\Phi))))$ is the truncated barycentre of Φ, and hence is its barycentre.

We have already seen that $\mathcal{EM}(V\mathrm{Stat}\ell^\infty) \simeq \mathbf{CCL}$, so composing with this equivalence gives $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCL}$. □

We can now show that every Eilenberg-Moore algebra of $\mathcal{E}$ is "observable", in the sense of [63].

Corollary 4.3.7. *Every $(X, \alpha) \in \mathrm{Obj}(\mathcal{EM}(\mathcal{E}))$ is observable, i.e. for each $x_1, x_2 \in X$, where $x_1 \neq x_2$, we have an $a : (X, \alpha) \to [0,1]$ in $\mathcal{EM}(\mathcal{E})$ such that $a(x_1) \neq a(x_2)$.*

Proof. By Theorem 4.3.6, we find $Y \in \mathrm{Obj}(\mathbf{CCL})$ such that $(X, \alpha) \cong (UY, \beta)$ in $\mathcal{EM}(\mathcal{E})$, where β is the Eilenberg-Moore algebra map for Y. Since $X \cong U(Y)$, $x_1, x_2 \in X$ map to $y_1, y_2 \in U(Y)$ which are still distinct. By Corollary 4.2.3 (iv) there exists a $\phi \in \mathbf{CCL}(Y, [0,1])$ such that $\phi(y_1) \neq \phi(y_2)$. Under the equivalence and composing with the isomorphism $X \cong UY$ we have a map $a : (X, \alpha) \to [0,1]$ in $\mathcal{EM}(\mathcal{E})$ such that $a(x_1) \neq a(x_2)$. □

4.4 Compact Effect Modules

Using Świrszcz's results, as shown in this chapter, we can characterize **CEMod** intrinsically, *i.e.* without using embedding in a vector space, in two different ways, according to whether we use $\mathcal{R}$ or $\mathcal{E}$.

We define the objects of $\mathbf{CEMod}_{\mathcal{R}}$ to be triples $(A, \mathcal{T}, \alpha_A)$, where A is an effect module, $\mathcal{T}$ a compact Hausdorff topology on A, and $\alpha_A : \mathcal{R}(A) \to A$ a map making A an Eilenberg-Moore algebra of $\mathcal{R}$, such that the $\mathcal{EM}(\mathcal{D})$-structures on A defined by α_A and the effect module structure of A are the same. The maps in $\mathbf{CEMod}_{\mathcal{R}}$ are maps that are effect module maps that are also continuous and $\mathcal{EM}(\mathcal{R})$ maps.

Similarly, the objects of $\mathbf{CEMod}_{\mathcal{E}}$ are pairs (A, α_A) where A is an effect module and $\alpha_A : \mathcal{E}(A) \to A$ is a map making A an Eilenberg-Moore algebra of $\mathcal{E}$ such that the $\mathcal{EM}(\mathcal{D})$-structures on A defined by α_A and the effect module structure of A are the same. The maps in $\mathbf{CEMod}_{\mathcal{E}}$ are effect module maps that are also $\mathcal{EM}(\mathcal{E})$ maps.

We write $U_{\mathcal{R}}$ for the comparison functor $\mathbf{CCL} \to \mathcal{EM}(\mathcal{R})$, known to be an equivalence by Theorem 4.2.9, and $U_{\mathcal{E}}$ for the comparison functor $\mathbf{CCL} \to \mathcal{EM}(\mathcal{E})$, an equivalence by Theorem 4.3.6.

We can therefore define functors $V_{\mathcal{R}} : \mathbf{CEMod} \to \mathbf{CEMod}_{\mathcal{R}}$ and $V_{\mathcal{E}} : \mathbf{CEMod} \to \mathbf{CEMod}_{\mathcal{E}}$ as follows. On objects $V_{\mathcal{R}}(E, A) = (A, \mathcal{T}, \alpha_A)$ where $\mathcal{T}$ is the subspace topology on A, and α_A the Eilenberg-Moore structure arising from $U_{\mathcal{R}}(E, A)$, considered as an object of **CCL**. This is an object of $\mathbf{CEMod}_{\mathcal{R}}$, because the condition that convex combinations of elements of A defined in E agree with those defined in terms of the effect module structure implies that the $\mathcal{EM}(\mathcal{D})$ structure defined by α_A and that defined by the effect module structure agree. On maps $V_{\mathcal{R}}$ does nothing, and any map that is affine and continuous is an $\mathcal{EM}(\mathcal{R})$ map, so this is well defined. The functor $V_{\mathcal{E}}$ is defined similarly, based on $U_{\mathcal{E}}$.

Proposition 4.4.1. *$V_{\mathcal{R}}$ and $V_{\mathcal{E}}$ are equivalences.*

Proof. As $V_{\mathcal{R}}$ and $V_{\mathcal{E}}$ are the identity on maps, they are both faithful. We can see that each functor is full as follows. If $g : V_{\mathcal{R}}(E, A) \to V_{\mathcal{R}}(F, B)$ is a map in $\mathbf{CEMod}_{\mathcal{R}}$, *i.e.* it is an effect module map $A \to B$ that is continuous and an $\mathcal{EM}(\mathcal{R})$ map. Then by the fullness of $U_{\mathcal{R}}$, it is an affine and continuous map $A \to B$, hence a morphism in **CEMod**. In fact, we did not use the fact that it is an $\mathcal{EM}(\mathcal{R})$-morphism and could have defined $\mathbf{CEMod}_{\mathcal{R}}$ without this condition. To see that $V_{\mathcal{E}}$ is full, let $g : V_{\mathcal{E}}(E, A) \to V_{\mathcal{E}}(F, B)$ be a map that is an effect module morphism and an $\mathcal{EM}(\mathcal{E})$-morphism. Then by the fullness of $U_{\mathcal{E}}$, g is a map $A \to B$ that is affine and continuous, considering A and B as subsets of E and F respectively, and therefore a morphism in **CEMod**.

We now consider essential surjectivity. If $(A, \mathcal{T}, \alpha_A)$ is an object of $\mathbf{CEMod}_{\mathcal{R}}$, by essential surjectivity of $U_{\mathcal{R}}$, there exists a locally convex space E, a compact convex subset X, and a map $i : A \to X$ that is an $\mathcal{EM}(\mathcal{R})$ isomorphism, where X is considered as an Eilenberg-Moore algebra of $\mathcal{R}$ using the barycentre map $\varepsilon_X : \mathcal{R}(X) \to X$. We can define an effect module structure on X so as to make this an isomorphism, and since every $\mathcal{EM}(\mathcal{R})$ map is $\mathcal{D}$-affine (Lemma 1.5.2), convex combinations defined using the effect module structure on X agree with convex combinations in E.

The proof of essential surjectivity for $V_{\mathcal{E}}$ is similar, using essential surjectivity of $U_{\mathcal{E}}$ and Proposition 1.5.8. □

4.5 Closing Remarks

In this chapter we saw two adjunctions and their composite, arranged as follows:

$$\begin{array}{c} \mathbf{CCL} \\ \acute{\mathcal{S}} \uparrow \dashv \downarrow U \\ \mathbf{CHaus} \\ \mathcal{U} \uparrow \dashv \downarrow W \\ \mathbf{Set}. \end{array}$$

The monad arising from the bottom adjunction was the ultrafilter monad, the monad arising from the top adjunction was the Radon monad, and the monad arising from the composite adjunction was the expectation monad, and all three adjunctions were monadic, giving rise to equivalences $\mathcal{EM}(\mathcal{U}) \simeq \mathbf{CHaus}$, $\mathcal{EM}(\mathcal{R}) \simeq \mathbf{CCL}$ and $\mathcal{EM}(\mathcal{E}) \simeq \mathbf{CCL}$, the first being due to Manes and the second two to Świrszcz. Perhaps more familiar in computer science is the following pair of adjunctions and their composite, from [134]

$$\begin{array}{c} \mathbf{CSL} \\ \uparrow \dashv \downarrow U \\ \mathbf{CHaus} \\ \mathcal{U} \uparrow \dashv \downarrow W \\ \mathbf{Set}, \end{array}$$

where **CSL** is the category of compact meet semilattices, or equivalently continuous lattices. The monad arising from the bottom adjunction is again the ultrafilter monad, the monad arising from the top adjunction is the Vietoris monad, and the monad arising from the composite adjunction is the *filter monad*. These adjunctions are all monadic again. This can be considered to be for nondeterminism what this chapter's results are for probability.

It might be interesting in future work to look at the relationship between these two situations and if there is a similar situation for the combination of probability and nondeterminism.

Chapter 5

W*-algebras and Measure Spaces

5.1 Introduction

From the viewpoint of noncommutative geometry, a C*-algebra is a noncommutative space, and a W*-algebra is a noncommutative measure space [22, 23]. This viewpoint, in analysis at least, started with the Koopman-von Neumann version of classical mechanics on Hilbert space [78, 131], involving only commuting operators[1], unlike the noncommuting operators required in quantum mechanics.

In the topological case, we have already seen the precise, categorical version of the statement that C*-algebras are noncommutative spaces, Gelfand duality. The functor $\mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ was C, taking the algebra of continuous $\mathbb{C}$-valued functions on a compact Hausdorff space, and the functor the other way was taking the spectrum of a commutative C*-algebra. By using non-unital C*-algebras one can also obtain a Gelfand duality for locally compact Hausdorff spaces.

The corresponding notions for measure spaces are not so clear. At least the following construction is well-known – if one has a σ-finite measure space (X, Σ, μ), one can take $L^\infty(X, \Sigma, \mu)$, the space of bounded measurable functions, identified if they agree outside a set of measure 0, and $L^\infty(X, \Sigma, \mu)$ is a commutative C*-algebra and the dual of $L^1(X, \Sigma, \mu)$, and therefore a W*-algebra. If one digs deeper, there is a relationship between a category whose objects are measure spaces where μ is a finite measure on a standard Borel space (X, Σ), and the opposite of the category of commutative W*-algebras with separable predual [85, Lecture 16, Corollary 10]. A similar category with measure-preserving maps is used in [122] to define conditional independence categorically. In the general case, Takesaki shows that every abelian von Neumann algebra is isomorphic to one of the form $L^\infty(X, \mathcal{B}o(X), \mu)$ where X is a locally compact

[1] except the unitaries representing time evolution

space, and μ a Radon measure [128, Theorem III.1.18]. Work in this area starts with Segal's work characterizing those measure spaces such that $L^\infty(X, \Sigma, \mu)$ is a commutative W*-algebra [119], and showing that Haar measures on locally compact groups satisfy this criterion, as well as disjoint unions of finite measure spaces, which would later be known as strictly localizable spaces. This is enough to get a rough idea of what category of measure spaces should be considered and that L^∞ will be essentially surjective. The fullness of L^∞ was essentially proved by Fremlin in [41], but in the case of complete Boolean algebras rather than commutative W*-algebras. Fremlin also deals with the lack of faithfulness, and particularly how badly it can fail. The main original contributions of this chapter are to define a suitable notion of morphism such that $L^\infty(f)$ is a normal morphism of W*-algebras, explicitly dealing with the quotient construction necessary to make L^∞ faithful, and patching together all the previous work into a categorical equivalence.

In summary, the purpose of this chapter is to define a category **Meas**, whose objects are measure spaces (of some restricted kind), the maps are classes of measurable maps (again with some restriction), such that L^∞ is a functor defining an equivalence

$$\mathbf{Meas} \simeq \mathbf{CW^*Alg}^{\mathrm{op}},$$

where $\mathbf{CW^*Alg}$ is the category having commutative W*-algebras as objects, and normal *-homomorphisms as maps.

There is also another duality, as observed by Heunen [52], using the work of Bezhanishvili [14, Corollary 6.10 (2)], ordinary Gelfand duality between compact Hausdorff spaces and commutative C*-algebras can be restricted to

$$\mathbf{HypStonean} \simeq \mathbf{CW^*Alg}^{\mathrm{op}}$$

where **HypStonean** is the category of hyperstonean spaces, the maps being continuous open maps. In some sense, **Meas** is the most general sort of measure space dual to $\mathbf{CW^*Alg}$, while **HypStonean** is a more canonical dual.

5.2 Preliminaries

5.2.1 Boolean Algebras

Recall that a Boolean algebra is a distributive lattice A in which every element a has a complement $\neg a$ [68, §I.1.6, p. 4]. A Boolean homomorphism is a function that preserves finite meets, finite joins and complements[2] As well as the lattice operations and complement, we may define the *difference*

$$a \setminus b = a \wedge \neg b,$$

and the *symmetric difference* operation

$$a \triangle b = (a \setminus b) \vee (b \setminus a).$$

[2] In fact, any two out of the three of these conditions suffices.

Recall that an ideal in a Boolean algebra A is a set I that is downward-closed and closed under binary joins. For each ideal, we can define a relation

$$a \sim b \Leftrightarrow a \triangle b \in I$$

for $a, b \in A$.

We first recall some basic facts about $\sim$ necessary for a good theory of quotient Boolean algebras.

Proposition 5.2.1.

(i) *If A and I and $a \sim b$ are as above, then*

$$a \sim b \Leftrightarrow a \setminus b \in I \text{ and } b \setminus a \in I,$$

agreeing with the definition in [121, §I.10].

(ii) *The relation $\sim$ is an equivalence relation.*

(iii) *If $a \sim a'$, then $\neg a \sim \neg a'$.*

(iv) *If $a \sim a'$ and $b \sim b'$, then $a \wedge b \sim a' \wedge b'$ and $a \vee b \sim a' \vee b'$.*

Proof.

(i) If $a \setminus b \in I$ and $b \setminus a \in I$, then $a \triangle b \in I$ because I is closed under union. If $a \triangle b \in I$ then $a \setminus b \in I$ because I is downward closed, and likewise for $b \setminus a$.

(ii) , (iii) and (iv) See [121, §I.10] or any basic text on Boolean algebras. □

The quotient Boolean algebra A/I is defined to be the set of $\sim$-equivalence classes. If $a \in A$, we write $[a] \in A/I$ for the equivalence class of a. The operations in A/I are defined so as to make $[\text{-}] : A \to A/I$ a Boolean homomorphism. We also show that the three natural ways of defining $[a] \leq [b]$ in a quotient Boolean algebra are equivalent.

Lemma 5.2.2. *Let A be a Boolean algebra, $I \subseteq A$ an ideal. Let $[a]$ represent the equivalence class of $a \in A$ as an element of A/I (equivalently, under $\sim$). The following three definitions of $[a] \leq [b]$ are equivalent.*

(i) *There exist $a', b' \in A$ such that $a \sim a'$, $b \sim b'$ and $a' \leq b'$.*

(ii) $a \setminus b \in I$

(iii) $[a] \wedge [b] = [a]$

The last is the usual definition of $\leq$ in Boolean algebras applied to A/I.

Proof.

- (i) $\Rightarrow$ (iii): Since $a \sim a'$ and $b \sim b'$, we have $a \wedge b \sim a' \wedge b'$ by Proposition 5.2.1 (iv). Then

  $$a \wedge b \sim a' \wedge b' = a' \sim a.$$

  Since $\sim$ is an equivalence relation (Proposition 5.2.1), we have $a \wedge b \sim a$. Therefore $[a \wedge b] = [a]$ and using Proposition 5.2.1 (iv) again, we have $[a] \wedge [b] = [a]$.

- (ii) $\Leftrightarrow$ (iii): We reason about $a \triangle (a \wedge b)$. We have

  $$\begin{aligned} a \triangle (a \wedge b) &= (a \wedge \neg(a \wedge b)) \vee (\neg a \wedge a \wedge b) = a \wedge (\neg a \vee \neg b) = a \wedge \neg b \\ &= a \setminus b. \end{aligned}$$

  We have arrived at the conclusion that $a \triangle (a \wedge b) \in I \Leftrightarrow a \setminus b \in I$. Since the left hand side of that is the definition of $a \sim (a \wedge b)$, we have $a \sim (a \wedge b) \Leftrightarrow a \setminus b \in I$.

- (iii) $\Rightarrow$ (i): From $[a] \wedge [b] = [a]$, we can deduce using Proposition 5.2.1 (iv) that $[a \wedge b] = [a]$ and so $a \wedge b \sim a$. So we take $a' = a \wedge b$ and $b' = b$. Then

  $$a' \wedge b' = (a \wedge b) \wedge b = a \wedge b = a',$$

  so $a' \leq b'$. □

A Boolean algebra is *complete* iff it is complete as a lattice, and complete Boolean algebra homomorphisms are those that preserve all joins and meets in addition to the Boolean operations. These definitions define a category **CBA**.

Lemma 5.2.3.

(i) *A poset isomorphism preserves (arbitrary) joins and meets.*

(ii) *A poset isomorphism between lattices preserves complements.*

(iii) *If A is a Boolean algebra and P a poset, $f : A \to P$ a poset isomorphism, then P is a Boolean algebra and f is an isomorphism of Boolean algebras. If A is complete, then P is complete, and the isomorphism is a morphism of complete Boolean algebras.*

(iv) *A bijective Boolean homomorphism is an isomorphism of Boolean algebras.*

Proof.

(i) An equivalence of categories preserves all limits and colimits present. Alternatively, consider a poset isomorphism $f : P \to Q$. Let $(x_i)_{i \in I}$ be a family of elements of P with least upper bound x. We want to show that

$$f(x) = \bigvee_{i \in I} f(x_i)$$

We have that $f(x_i) \leq f(x)$ for all $i \in I$ by monotonocity of f. Therefore we only need show it is a the least of all upper bounds. So let $y \in Q$ be an upper bound of $f(x_i)$. By monotonicity of the inverse, we have $x_i \leq f^{-1}(y)$, and so $x \leq f^{-1}(y)$. Applying f, we find $f(x) \leq y$, as required.

The same is true for greatest lower bounds by viewing f as a map $P^{\mathrm{op}} \to Q^{\mathrm{op}}$ and reusing the previous argument.

(ii) Recall that a complement of $x \in P$ is an element $y \in P$ such that $x \wedge y = 0$ and $x \vee y = 1$. We show that if y is a complement of x, then $f(y)$ is a complement of $f(x)$. To see this we actually only need the previous result that such an isomorphism preserves joins and meets, so

$$f(x) \wedge f(y) = f(x \wedge y) = f(0) = 0$$

and

$$f(x) \vee f(y) = f(x \vee y) = f(1) = 1$$

preservation of 0 and 1 occuring because they are the join and meet of $\emptyset$, respectively.

(iii) Being a Boolean algebra or a complete Boolean algebra is a property of a poset, rather than a structure. This is because being a lattice, or a complete lattice, is a property as it can be defined in terms of suprema and infima, being distributive is a property, having complements in the sense of (ii) is a property, and distributivity guarantees that these complements are unique (see [68, §I.1.6] or [16, Chapter X, Theorem 1]). We show that a poset isomorphism $f : A \to P$ makes P have these properties if A does. We know from (i) that P must be a lattice if A is and a complete lattice if A is complete, as any for any set $S \subseteq P$ we want to take the supremum of, we can take the supremum of $f^{-1}(S)$, and take the image of the resulting element. The same holds for infima, and this also shows that f is a lattice homomorphism, or a complete lattice homomorphism in the relevant case.

We show distributivity as follows. We want to show that for all $x, y, z \in P$, that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We have

$$\begin{aligned} f^{-1}(x \wedge (y \vee z)) &= f^{-1}(x) \wedge (f^{-1}(y) \vee f^{-1}(z)) && \text{by (i)} \\ &= (f^{-1}(x) \wedge f^{-1}(y)) \vee (f^{-1}(x) \wedge f^{-1}(z)) && A \text{ is Boolean} \\ &= f^{-1}(x \wedge y) \vee f^{-1}(x \wedge z) \\ &= f^{-1}((x \wedge y) \vee (x \wedge z)). \end{aligned}$$

Then $\vee$ distributes over $\wedge$ by a similar argument.

Using (ii), we can show that complements exist, because if we have $x \in P$, then $f(\neg f^{-1}(x))$ is a complement for x, and hence is *the* complement by the distributivity. This is enough to make P a Boolean algebra. This shows also that f preserves complements, and therefore is a Boolean algebra homomorphism. If A is complete, f is a morphism of complete Boolean algebras because it is a morphism of complete lattices in this case.

(iv) Let $f : A \to B$ be a bijective Boolean homomorphism, and g its inverse. For any elements $a, b \in B$ we have

$$g(a \wedge b) = g(f(g(a)) \wedge f(g(b))) = g(f(g(a) \wedge g(b))) = g(a) \wedge g(b)$$

and

$$g(\neg a) = g(\neg f(g(a))) = g(f(\neg g(a))) = \neg g(a)$$

so g is a Boolean homomorphism as well, and so f is an isomorphism. □

We must emphasize that although an orthocomplement in an orthomodular lattice is a complement in the above sense, lattice isomorphisms of non-distributive orthomodular lattices need not preserve orthocomplements, as complements are not necessarily unique and so need not coincide with the chosen orthocomplement. In other words, an orthocomplement is a structure and not just a property for non-distributive lattices.

Note that since $\neg : A \to A^{\mathrm{op}}$ is a poset isomorphism by the definition of a Boolean algebra, the previous lemma implies that the infinitary version of de Morgan's law holds in a complete Boolean algebra. Since left adjoints preserve limits, this also shows that finite meets distribute over infinite joins, and this shows that finite joins distribute over infinite meets also.

We take the opportunity to prove the following theorem about families of subsets.

Lemma 5.2.4. *Let $\{X_i\}_{i\in I}, \{Y_i\}_{i\in I}$ be two families of subsets of some set, such that each family is pairwise disjoint and $X_i \cap Y_j$ is non-empty only if $i = j$. Then*

$$\left(\bigcup_{i\in I} X_i\right) \triangle \left(\bigcup_{i\in I} Y_i\right) = \bigcup_{i\in I} X_i \triangle Y_i$$

Proof. Given the definition $X \triangle Y = X \setminus Y \cup Y \setminus X$, this will follow from

$$\left(\bigcup_{i\in I} X_i\right) \setminus \left(\bigcup_{i\in I} Y_i\right) = \bigcup_{i\in I} X_i \setminus Y_i$$

by symmetry and the fact that unions distribute over unions. So we reduce to proving the above statement.

Suppose x is an element of the left hand side. Then there is some $k \in I$ such that $x \in X_k$ and x is not in Y_i for any $i \in I$. Therefore $x \in X_k \setminus Y_k$ and so x is in the right hand side.

Suppose now that x is an element of the right hand side. Then there is some $k \in I$ such that $x \in X_k \setminus Y_k$. But $x \in X_k$ implies $x \notin Y_i$ for any $i \neq k$, so we just have $x \notin \bigcup_{i\in I} Y_i$, and hence x is in the left hand side. □

5.2.2 General Topology

We begin with some basic lemmas on closure and nowhere dense sets that hold in any topological space. Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior.

Lemma 5.2.5. *Let U be an open set in a space X. Then $\mathrm{cl}\,(U) \setminus U$ is nowhere dense.*

Proof. We have that

$$\mathrm{cl}\,(U) \setminus U = \mathrm{cl}\,(U) \cap (X \setminus U)$$

and is therefore the intersection of two closed sets, and hence closed. So it is nowhere dense iff it has empty interior. Suppose there were a non-empty open set $U' \subseteq (\mathrm{cl}\,(U) \setminus U)$. Then $\mathrm{cl}\,(U) \setminus U'$ would be a closed set containing U that is strictly smaller than $\mathrm{cl}\,(U)$, a contradiction. □

Lemma 5.2.6. *Let X be a topological space.*

(i) If U is an open subspace and $S \subseteq U$, S is U-open iff it is X-open.

(ii) If C is a closed subspace and $S \subseteq C$, S is C-closed iff it is X-closed.

Proof.

(i) Suppose S is X-open. Then $S = S \cap U$ and hence is U-open. For the other direction, suppose S is U-open. There exists $S' \subseteq X$ that is X-open such that $S' \cap U = S$. As a finite intersection of X-open sets, S is X-open.

(ii) Suppose S is X-closed. Then $C = S \cap C$ and hence is C-closed. For the other direction, suppose S is C-closed. There exists $S' \subseteq X$ that is closed such that $S' \cap C = S$. As an intersection of X-closed sets, S is X-closed.

□

Lemma 5.2.7. *Let X be a topological space, G a clopen subset and S a subset of G. The closure and interior of S in the space G agree with the closure and interior of S in X.*

Proof.

- The closure operator:

  We define
  $$C_G = \{C \mid S \subseteq C \subseteq G,\ \text{C closed in G}\}$$
  and
  $$C_X = \{C \mid S \subseteq C \subseteq X,\ \text{C closed in X}\}$$
  and we have that
  $$\mathrm{cl}_G\,(S) = \bigcap C_G$$

and

$$\mathrm{cl}_X(S) = \bigcap C_X.$$

If $x \in \mathrm{cl}_G(S)$, then $x \in C$ for all $C \in C_G$. If $C \in C_X$, then $C \cap G$ is closed, and therefore an element of C_G as well as a subset of C, so $x \in C$. Therefore $x \in C$ for all $C \in C_X$ and $\mathrm{cl}_G(S) \subseteq \mathrm{cl}_X(S)$. On the other hand, if $x \in \mathrm{cl}_X(S)$, then $x \in C$ for all $C \in C_X$. If $C \in C_G$, we have a closed set C' in X such that $C' \cap G = C$, so C is closed in X, and therefore $C \in C_X$, so $x \in C$. Therefore $x \in C$ for all $C \in C_G$, so $\mathrm{cl}_X(S) \subseteq \mathrm{cl}_G(S)$. Therefore the two sets are equal.

- The interior operator:

  We define

  $$U_G = \{U \mid U \subseteq S, \text{ U open in G}\}$$

  and

  $$U_X = \{U \mid U \subseteq S, \text{ U open in X}\}$$

  and we have that

  $$\mathrm{int}_G(S) = \bigcup U_G$$

  and

  $$\mathrm{int}_X(S) = \bigcup U_X$$

  If $U \in U_X$, then $U \cap G = U$ and so U is open in G and hence $U \in U_G$. If $U \in U_G$, then there is some $U' \subseteq X$ that is open such that $U' \cap G = U$. Since G is open, $U' \cap G$ is open, being the intersection of two open sets, so U is actually open in X, hence $U \in U_X$. Therefore $U_G = U_X$ and hence $\mathrm{int}_G(S) = \mathrm{int}_X(S)$. □

Lemma 5.2.8. *Let X_i be an I-indexed family of disjoint clopen subsets of a space Y. Let $N_i \subseteq X_i$ be a family of nowhere dense sets. Then*

$$N = \bigcup_{i \in I} N_i$$

is nowhere dense.

Proof. For ease of use later, we define

$$X = \bigcup_{i \in I} X_i.$$

First, we can redefine N_i to be the closures of the sets, since these are still subsets of X_i since X_i is closed. So we take N_i to be a family of closed sets of empty interior. Suppose for a contradiction that $\mathrm{cl}(N)$ contains a non-empty open set U. If $x \in \mathrm{cl}(N) \cap X_i$, then since

$$\begin{aligned}\mathrm{cl}(N) &= \mathrm{cl}(N_i \cup (N \setminus N_i)) \\ &= \mathrm{cl}(N_i) \cup \mathrm{cl}(N \setminus N_i) \\ &= N_i \cup \mathrm{cl}(N \setminus N_i)\end{aligned}$$

and $N \setminus N_i \subseteq Y \setminus X_i$, which is a closed set, we have that $x \in N_i$, and so we have shown that $\mathrm{cl}\,(N) \cap X_i = N_i$.

If $U \cap X_i$ were non-empty, it would therefore be a non-empty open subset of N_i, which contradicts N_i having empty interior. Therefore $U \cap X_i = \emptyset$, and so $U \cap X = \emptyset$. Since $N_i \subseteq X_i$, we have that $N \subseteq X$ and consequently $\mathrm{cl}\,(N) \subseteq \mathrm{cl}\,(X)$. The set U was chosen to be a subset of $\mathrm{cl}\,(N)$, which, when combined with our previous remark means it is contained in $\mathrm{cl}\,(X) \setminus X$, but as this is the difference of an open set and its closure, $\mathrm{cl}\,(X) \setminus X$ has empty interior (Lemma 5.2.5), a contradiction. Therefore U cannot exist and N is nowhere dense. □

Recall that a σ-ideal on a set X is an ideal in $\mathcal{P}(X)$ that is also closed under countable joins (countable unions). The meagre sets in a topological space X are the elements of the σ-ideal generated by the nowhere dense sets. We use $\mathcal{M}eagre(X)$ as the symbol for this σ-ideal. Equivalently, a subset of X is meagre if it is a countable union of nowhere dense sets. A set S in a topological space X has the *Baire property* if there is an open set U such that $S \triangle U$ is a meagre set. Recall that a σ-algebra $\Sigma \subseteq \mathcal{P}(X)$ is a family of sets closed under countable Boolean operations, *i.e.* complements, countable unions and countable intersections. A set is *Borel* if it is in the σ-algebra generated by the open sets. We use $\mathcal{B}o(X)$ to mean the σ-algebra of Borel sets of the topological space X.

Proposition 5.2.9. *The collection of sets having the Baire property is a σ-algebra containing the open sets. Thus every Borel set has the Baire property.*

Proof. An open set has the Baire property because the empty set is nowhere dense. We therefore move on to showing that sets with the Baire property are a σ-algebra. By distributivity, it suffices to show that sets with the Baire property are closed under complement and countable union.

- Closure under complement: Suppose $S \subseteq X$ has the Baire property, *i.e.* that there exists an open set U such that $S \triangle U$ is meagre. We want to show that $X \setminus S$ has the Baire property. By Lemma 5.2.5, $X \setminus U$ has the Baire property, differing from its interior by a nowhere dense set. Therefore we are done with this part if $(X \setminus S) \triangle (X \setminus U)$ is meagre, using Proposition 5.2.1(ii). For ease of notation, we use the Boolean algebra notation for complements:

$$\begin{aligned}\neg S \triangle \neg U &= \neg S \setminus \neg U \cup \neg U \setminus \neg S = (\neg S \cap \neg\neg U) \cup (\neg U \cap \neg\neg S)\\ &= (U \cap \neg S) \cup (S \cap \neg U) = U \setminus S \cup S \setminus U = S \triangle U,\end{aligned}$$

  which was assumed to be meagre.

- Closure under countable union: Suppose $(S_i)_{i\in\mathbb{N}}$ have the Baire property, *i.e.* there exists $(U_i)_{i\in\mathbb{N}}$, with each U_i open, such that for all $i \in \mathbb{N}$, $S_i \triangle U_i$

is meagre. Then

$$\left(\bigcup_{i=1}^{\infty} S_i\right) \triangle \left(\bigcup_{j=1}^{\infty} U_j\right) = \left(\bigcup_{i=1}^{\infty} S_i \setminus \bigcup_{j=1}^{\infty} U_j\right) \cup \left(\bigcup_{i=1}^{\infty} U_i \setminus \bigcup_{j=1}^{\infty} S_j\right)$$
$$= \bigcup_{i=1}^{\infty} \left(S_i \setminus \bigcup_{j=1}^{\infty} U_j\right) \cup \bigcup_{i=1}^{\infty} \left(U_i \setminus \bigcup_{j=1}^{\infty} S_j\right),$$

using distributivity. Now,

$$S_i \setminus \bigcup_{j=1}^{\infty} U_j \subseteq S_i \setminus U_i$$

and so is a meagre set, being a subset of $S_i \triangle U_i$. Therefore the union over all $i \in \mathbb{N}$ is also meagre, as meagre sets are a σ-ideal. A similar argument works for the other side, and this shows that the set difference is meagre. Since $\bigcup_{i=1}^{\infty} U_i$ is an open set, we can conclude that $\bigcup_{i=1}^{\infty} S_i$ has the Baire property. □

Lemma 5.2.10. *If $\{S_n\}_{n\in\mathbb{N}}$ is a sequence of sets, $S_n \in \Sigma$, where Σ is some σ-algebra. Then there is a sequence of sets $\{T_n\}_{n\in\mathbb{N}}$ that is pairwise disjoint such that $\bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} T_n$ and $T_n \subseteq S_n$.*

Proof. Define $T_n = S_n \setminus \bigcup_{i=1}^{n-1} S_i$. It is clear that $T_n \subseteq S_n$. Suppose $n, m \in \mathbb{N}$ and $n \neq m$. Without loss of generality take $n < m$. Then, since $T_n \subseteq S_n$ and $S_n \cap T_m = \emptyset$ since it was part of the union that was subtracted, we have $T_n \cap T_m = \emptyset$. This shows the sequence is pairwise disjoint. Now, if $x \in \bigcup_{n=1}^{\infty} T_n$, then $x \in S_n$ for some n, and so $x \in \bigcup_{n=1}^{\infty} S_n$. If $x \in \bigcup_{n=1}^{\infty} S_n$, by the well-orderedness of $\mathbb{N}$, there is a smallest n such that $x \in S_n$. For this n, we have $x \in S_n \setminus \bigcup_{i=1}^{n-1} S_i$, and so $x \in T_n$. This shows that the two sequences have the same union. □

5.2.3 Compact Hausdorff Spaces, Stone Spaces and C*-algebras

A Stone space X is a compact Hausdorff space that is 0-dimensional, *i.e.* the family of clopen sets in X is a base. One of the basic examples of a categorical duality is *Stone duality*, relating the category of Boolean algebras **BA**, and the category of Stone spaces **Stone**. We define, for X a Stone space and A a Boolean algebra:

$$\begin{aligned}
\text{Clopen}(X) &= \{G \subseteq X \mid G \text{ clopen}\} \\
\text{Clopen}(f : X \to Y)(G) &= f^{-1}(G) \\
\text{Spec}(A) &= \{\phi : A \to 2 \mid \phi \text{ Boolean}\} \\
\text{Spec}(g : A \to B)(\phi) &= \phi \circ g,
\end{aligned}$$

with the topology on Spec(A) being generated by the basic clopens

$$G_a = \{\phi : A \to 2 \mid \phi(a) = 1\}$$

for $a \in A$.

Theorem 5.2.11 (Stone Duality)**.** *The above definitions define functors*

$$\text{Clopen} : \mathbf{Stone} \to \mathbf{BA}^{\text{op}}$$
$$\text{Spec} : \mathbf{BA}^{\text{op}} \to \mathbf{Stone}$$

and make an adjoint equivalence $\mathbf{BA}^{\text{op}} \simeq \mathbf{Stone}$ *with the following unit and counit*

$$\eta_X : X \to \text{Spec}(\text{Clopen}(X)) \qquad \epsilon_A : \text{Clopen}(\text{Spec}(A)) \leftarrow A$$
$$\eta_X(x)(G) = \begin{cases} 1 & \textit{if } x \in G \\ 0 & \textit{if } x \notin G \end{cases} \qquad \epsilon_A(a) = \{\phi \in \text{Spec}(A) \mid \phi(a) = 1\}$$

Since η and ϵ are generic names for the unit and counit of an adjunction, when ambiguity might arise we use the names $\eta^{\mathbf{BA}}$ and $\epsilon^{\mathbf{BA}}$. The functor Clopen can be defined more generally to be $\mathbf{Top} \to \mathbf{BA}^{\text{op}}$ and we will sometimes take it to be defined as such. In this case Stone duality is merely an adjunction. Similarly, we refer to the unit and counit of Gelfand duality (Theorem 1.2.4) as $\eta^{\mathbf{C}^*}$ and $\epsilon^{\mathbf{C}^*}$ when necessary.

Recall the following facts: A *self-adjoint* element of a C*-algebra is an element a such that $a^* = a$. A *positive* element p in a C*-algebra is one that can be expressed as $p = a^*a$ for some a. A map between C*-algebras $A \to B$ is *positive* if it maps positive elements to positive elements. This is equivalent to monotonicity in the partial orders induced by the positive cones. There is also a category $\mathbf{C^*Alg}_{\text{PU}}$ of C*-algebras and positive unital maps. A *state* on a C* algebra A is a positive unital map $A \to \mathbb{C}$.

A *projection* in a C*-algebra A is a self-adjoint idempotent element. Using the ordering A given by the positive cone, projections form a poset, which we write as Proj(A). We can define Proj on *-homomorphisms as

$$\text{Proj}(f)(p) = f(p)$$

because *-homomorphisms preserve self-adjoint elements and idempotent elements, as they preserve -* and the multiplication. Since *-homomorphisms are positive, the map Proj(f) is monotone. The argument that this defines a functor $\mathbf{C^*Alg} \to \mathbf{Poset}$ is trivial, where **Poset** is the category of partially ordered sets and monotone maps.

Given a projection p, we define $\neg p$ to mean $1 - p$.

Lemma 5.2.12. *Let* p, q *be projections in a C*-algebra* A.

(i)
$$p \leq q \Leftrightarrow qp = p \Leftrightarrow pq = p \Leftrightarrow q - p \in \text{Proj}(A)$$

(ii) *If* $p \perp q$ *means* $p \leq \neg q$, *then*

$$p \perp q \Leftrightarrow qp = 0 \Leftrightarrow pq = 0$$

(iii) *If* p *and* q *commute, then* $p \wedge q$ *exists in* $\mathrm{Proj}(A)$ *and* $p \wedge q = pq$.

(iv) *If* p *and* q *commute,* $p \vee q$ *exists in* $\mathrm{Proj}(A)$ *and it equals* $p + q - p \wedge q = p + q - pq$.

Proof.

(i) We prove this by the following implications:

- $p \leq q \Rightarrow qp = p$: The following is based on [116, Lemma 1.2]. We have that $(1-q)p(1-q) \leq (1-q)q(1-q) = 0$. We then see that

$$\|(1-q)p\|^2 = \|(1-q)pp^*(1-q)^*\| = \|(1-q)p(1-q)\| = 0.$$

  Therefore $(1-q)p = 0$, so

$$p = ((1-q)+q)p = (1-q)p + qp = qp.$$

- $pq = p \Leftrightarrow qp = p$:
  This is shown as follows

$$pq = p \Leftrightarrow (pq)^* = p^* \Leftrightarrow q^*p^* = p \Leftrightarrow qp = p.$$

- $pq = p$ and $qp = p$ implies $q - p \in \mathrm{Proj}(A)$:
  We have that $q - p$ is self-adjoint, so we only need show that it is idempotent:

$$(q-p)^2 = q^2 - pq - qp + p^2 = q - p - p + p = q - p.$$

- $q - p \in \mathrm{Proj}(A) \Rightarrow p \leq q$: This is true because every projection is positive.

(ii) The equivalence between $pq = 0$ and $qp = 0$ is proved using the self-adjointness of p and q in the same manner that was used to prove $pq = p \Leftrightarrow qp = p$ in part (i). To show that $p \leq 1 - q$ implies $pq = 0$, we start by using (i) to deduce that $p(1-q) = p$. Since $p(1-q) = p - pq$, we have $p - pq = p$, which shows that $pq = 0$. For the opposite implication, suppose that $pq = 0$. Then $p(1-q) = p - pq = p$, so by (i) we have $p \leq (1-q)$.

(iii) We need to show that if p and q commute, pq is a projection and if r is a projection such that $r \leq p$ and $r \leq q$, then $r \leq pq$. For the first statement, we show pq is self adjoint by observing $(pq)^* = q^*p^* = qp = pq$, using the fact that p and q commute. To show idempotence, $(pq)^2 = pqpq = ppqq = pq$. To show that $pq = p \wedge q$, we first see that by (i), we have that $rp = r$ and $rq = r$, so $r(pq) = rq = r$. Using (i) again, this shows that $r \leq pq$.

(iv) We use that $\neg : \mathrm{Proj}(A)^{\mathrm{op}} \to \mathrm{Proj}(A)$ is an isomorphism of posets, so by Lemma 5.2.3 it preserves the lattice operations, and so we have $p \vee q = \neg(\neg p \wedge \neg q)$, or de Morgan's law. To apply part (iii), we will show that if p and q commute, then $(1-p)$ and $(1-q)$ commute. We start with

$$(1-p)(1-q) = 1-p-q+pq = 1-q-p+qp = (1-q)(1-p).$$

We can now apply part (iii) to get

$$\begin{aligned} p \vee q &= 1-((1-p)\wedge(1-q)) = 1-((1-p)(1-q)) \\ &= 1-(1-p-q+pq) = p+q-pq = p+q-p\wedge q \end{aligned}$$

as required. □

In commutative C*-algebras, projections have an important characterization.

Lemma 5.2.13. *Let X be a compact Hausdorff space. The following pair of maps is an order isomorphism between* $\mathrm{Proj}(C(X))$ *and* $\mathrm{Clopen}(X)$*.*

$$s : \mathrm{Proj}(C(X)) \to \mathrm{Clopen}(X) \qquad c : \mathrm{Clopen}(X) \to \mathrm{Proj}(C(X))$$
$$s(P) = P^{-1}(1) \qquad c(S) = \chi_S$$

Proof. First we show that the maps are defined correctly. If $P \in C(X)$ is a projection, we have that $\forall x \in X.P(x)^2 = P(x)$. The only complex numbers for which this is satisfied are 0 and 1, so P is $\{0,1\}$-valued. We therefore have that $s(P) = P^{-1}(1) = P^{-1}(X \setminus \{0\})$, and since $s(P)$ is the preimage of both a closed set and an open set, $s(P)$ is clopen.

If S is a clopen set, we have a function χ_S that we must show is continuous. There are four cases. Suppose U is an open set:

(i) If $\{0,1\} \not\subseteq U$, then $\chi_S^{-1}(U) = \emptyset$ which is open.

(ii) If $\{0,1\} \subseteq U$ then $\chi_S^{-1}(U) = X$ which is open.

(iii) If $1 \in U$ and $0 \notin U$, we have that $\chi_S^{-1}(U) = \chi_S^{-1}(1)$, which is open because it is clopen.

(iv) If $0 \in U$ and $1 \notin U$, we have that $\chi_S^{-1}(U) = \chi_S^{-1}(0) = X \setminus S$ which is open because it is the complement of the closed set S.

It is then a projection because 1 and 0 are real and idempotent in $\mathbb{C}$, and so pointwise multiplication retains these properties.

We show that s and c are monotone. Suppose we have two clopen sets and $S_1 \subseteq S_2$. We have that $\chi_{S_1}(x) = 1$ implies $\chi_{S_2}(x) = 1$. Therefore $\chi_{S_1}(x) \leq \chi_{S_2}(x)$ for all $x \in X$, so c is monotone. Now suppose $P_1 \leq P_2$. We have that if $P_1(x) = 1$ then $P_2(x) = 1$, so $P_1^{-1}(1) \subseteq P_2^{-1}(1)$ and hence s is monotone.

To show that s and c are mutually inverse, let S be a clopen of X. Then

$$s(c(S)) = s(\chi_S) = \chi_S^{-1}(1) = S.$$

Let P be a projection in $C(X)$. Then

$$c(s(P)) = c(P^{-1}(1)) = \chi_{P^{-1}(1)}$$

Let $x \in X$. We already know that $P(x) \in \{0, 1\}$, so we reduce to these two cases.

(i) If $P(x) = 1$, then $x \in P^{-1}(1)$, so $\chi_{P^{-1}(1)}(x) = 1$.

(ii) If $P(x) = 0$, then $x \notin P^{-1}(1)$, so $\chi_{P^{-1}(1)}(x) = 0$.

hence $P = \chi_{P^{-1}(1)}$ and so $c(s(P)) = P$. □

Corollary 5.2.14. *If A is a commutative C^*-algebra, we have an order isomorphism between* Proj(A) *and* Clopen(Spec(A))*. When restricted to commutative C^*-algebras,* Proj *is a functor* $\mathbf{CC^*Alg} \to \mathbf{BA}$.

Proof. For the first part, apply Gelfand duality to Lemma 5.2.13. In detail, the map $\mathrm{Proj}(A) \to \mathrm{Clopen}(\mathrm{Spec}(A))$ is given by $s \circ \mathrm{Proj}(\epsilon_A)$, and the map $\mathrm{Clopen}(\mathrm{Spec}(A)) \to \mathrm{Proj}(A)$ is given by $\mathrm{Proj}(\epsilon_A^{-1}) \circ c$.

Since Clopen(Spec(A)) is a Boolean algebra, by Lemma 5.2.3(iii) Proj(A) is a Boolean algebra if A is commutative. We therefore only need to show that for $f : A \to B$ a *-homomorphism, Proj(f) preserves lattice operations, as this implies that the complements are preserved for Boolean algebras. By Lemma 5.2.12(iii) and (iv), for any pair of projections p, q, $p \wedge q = pq$ and $p \vee q = p + q - pq$. Therefore these operations are preserved by any *-homomorphism. □

Lemma 5.2.15.

(i) If $f : X \to Y$ is a map of sets, $S \subseteq Y$, then $\chi_S \circ f = \chi_{f^{-1}(S)}$.

(ii) Let $f : X \to Y$ be a continuous map of topological spaces and G a clopen set (so that χ_G is continuous). Then $C(f)(\chi_G) = \chi_{f^{-1}(G)}$.

(iii) The isomorphisms $c : \mathrm{Clopen}(X) \to \mathrm{Proj}(C(X))$ from Lemma 5.2.13 form a natural transformation $c : \mathrm{Clopen} \Rightarrow \mathrm{Proj} \circ C$ (as do the maps s).

Proof.

(i) We see that $\chi_S(f(x)) = 1$ if $f(x) \in S$ and 0 if $f(x) \notin S$. Now, $f(x) \in S$ iff $x \in f^{-1}(S)$, by definition, so if $\chi_S(f(x)) = 1$, we have that $\chi_{f^{-1}(S)}(x) = 1$ and likewise for 0.

(ii) Expanding the definitions, $C(f)(\chi_G) = \chi_G \circ f = \chi_{f^{-1}(G)}$ using (i).

(iii) The naturality diagram, for a continuous function $f : X \to Y$ is

$$\begin{array}{ccc} \mathrm{Clopen}(Y) & \xrightarrow{c_Y} & \mathrm{Proj}(C(Y)) \\ {\scriptstyle \mathrm{Clopen}(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{Proj}(C(f))} \\ \mathrm{Clopen}(X) & \xrightarrow[c_X]{} & \mathrm{Proj}(C(X)). \end{array}$$

We want to show that this commutes, *i.e.* that $c_X \circ \text{Clopen}(f) = \text{Proj}(C(f)) \circ c_Y$. So let $G \in \text{Clopen}(Y)$, and we see that the upper right path gives

$$\text{Proj}(C(f))(c_Y(G)) = \text{Proj}(C(f))(\chi_G) = C(f)(\chi_G).$$

The lower left path reduces to

$$c_Y(\text{Clopen}(f)(G)) = c_X(f^{-1}(G)) = \chi_{f^{-1}(G)}.$$

In (ii) we proved that these are equal, so the diagram commutes. □

We have the following useful lemma about clopen subsets of compact Hausdorff spaces.

Lemma 5.2.16. *Let X be a compact Hausdorff space and G a clopen subset. Define*

$$\gamma : C(G) \to C(X)$$
$$\gamma(f)(x) = \begin{cases} 0 & \text{if } x \notin G \\ f(x) & \text{if } x \in G \end{cases}.$$

This map is linear and preserves products and *, hence is positive. It also preserves bounded directed suprema whenever they exist.

Proof. First we show that $\gamma(f)$ is continous on X. Let U be an open subset of X. If $0 \notin U$, we have

$$\gamma(f)^{-1}(U) = f^{-1}(U)$$

which is an open subset of G. Since G has the subspace topology, there is some V such that $V \cap G = U$. But since G is open, we have that U is already open as the intersection of two open sets.

Now suppose that $0 \in U$. We have then that

$$\gamma(f)^{-1}(U) = f^{-1}(U) \cup X \setminus G.$$

By the same reasoning as the previous case, $f^{-1}(U)$ is open, and $X \setminus G$ is open because G is closed, so $\gamma(f)^{-1}(U)$ is open. Thus $\gamma(f) \in C(X)$.

Preservation of addition, scalar multiplication, multiplication and * hold because the definitions of these operations are pointwise. Positivity is implied because if $f \in C(G)$ is positive, then $f = g^*g$ for some $g \in C(G)$. Then $\gamma(f) = \gamma(g)^*\gamma(g)$ and hence is positive.

Finally we show that it preserves bounded directed suprema if they exist. Let f_α be a directed subset of $C(G)_+$ with supremum f. Since f is positive, it is monotone and so $\gamma(f) \geq \gamma(f_\alpha)$ for all α, so the property of being an upper bound is preserved. Now suppose g is an upper bound for $\gamma(f_\alpha)$. We have that $g|_G$ is an upper bound for f_α and so $f \leq g|_G$, i.e. $\forall x \in G. f(x) \leq g(x)$. From the fact that g is an upper bound for $\gamma(f_\alpha)$, we have that $\forall x \in X \setminus G. g(x) \geq \gamma(f_\alpha)(x) = 0 = \gamma(f)(x)$. Putting these two pieces together, we have that $g \geq \gamma(f)$, so $\gamma(f)$ is the least upper bound. □

It is apparent that λ does not preserve the unit unless $G = X$.

5.2.4 Boolean Algebras and POVMs

Let E be a Banach order-unit space. The unit interval $[0,1]_E$ is a Banach effect module, and so we can consider the hom set $\mathbf{EA}(A,[0,1]_E)$ for any Boolean algebra A. Elements of this hom set are called *POVMs*, by analogy to the case when $E = B(\mathcal{H})$, where POVM stands for Positive Operator-Valued Measure. A POVM can equivalently be defined to be a function $\mu : A \to E_+$ such that $\mu(1) = 1$ and if $a \wedge b = 0$, $\mu(a \vee b) = \mu(a) + \mu(b)$. This implies that the image of μ is contained in $[0,1]_E$. If E is a C*-algebra whose projections form a lattice, we can define *PVMs*, or Projection-Valued Measures, as lattice homomorphisms $A \to \mathrm{Proj}(E)$.

To do this, we define *simple functions* on X, for X a Stone space. We take

$$\mathrm{Simp}(X,k) = \{a : X \to k \mid a \text{ continuous, with finite range}\}$$

where k is either $\mathbb{R}$ or $\mathbb{C}$.

We can define $\mathrm{Simp}(\text{-},k)$ on continuous maps as the restriction of C, by precomposition, making a contravariant functor. This is because if $a : Y \to k$ is continuous with finite range and $f : X \to Y$ is continuous, then $a \circ f$ is continuous with finite range, and the rest of the proof of functoriality follows from the functoriality of C.

If $\{\alpha_i\}_{i\in I}$, I a finite set, is the set of values of a simple function a, then we can express a as

$$\sum_{i\in I} \alpha_i \chi_{G_i},$$

where $G_i = f^{-1}(\{\alpha_i\})$. This representation is unique, in the following sense.

Lemma 5.2.17.

(i) Let X be a Stone space. Every $a \in \mathrm{Simp}(X,k)$ can be expressed as

$$\sum_{i\in I} \alpha_i \chi_{G_i}$$

where I is a finite set, $(\alpha_i)_{i\in I}$ a family of distinct numbers in k, and $(G_i)_{i\in I}$ a family of disjoint clopen sets in X. This is called the reduced form *of a.*

(ii) Let $(\alpha_i)_{i\in I}$ be a finite family of distinct numbers in k and (S_i) a disjoint family of subsets of a set X, and $(\beta_j)_{j\in J}$ and $(T_j)_{j\in J}$ likewise. Suppose

$$\sum_{i\in I} \alpha_i \chi_{S_i} = \sum_{j\in J} \beta_j \chi_{T_j}.$$

Then there is a bijection $s : I \to J$ such that $\alpha_i = \beta_{s(i)}$ and $S_i = T_{s(i)}$.

Proof.

(i) We take I to be any set with the same cardinality as the range of a, and define (α_i) to be the values, which are necessarily distinct. We define $G_i = a^{-1}(\{\alpha_i\})$. If $i \neq j \in I$, then $\{\alpha_i\} \cap \{\alpha_j\} = \emptyset$, so

$$\begin{aligned} G_i \cap G_j &= a^{-1}(\{\alpha_i\}) \cap a^{-1}(\{\alpha_j\}) \\ &= a^{-1}(\{\alpha_i\} \cap \{\alpha_j\}) \\ &= a^{-1}(\emptyset) = \emptyset, \end{aligned}$$

showing that it is a disjoint family. By the continuity of a, all the sets G_i are closed, so we are left to prove that they are also open. Define $U_i = \bigcap_{j \in I \setminus \{i\}} k \setminus \{\alpha_j\}$. As a finite intersection of open sets, each U_i is open. By the continuity of a, $a^{-1}(U_i)$ is open, and

$$\begin{aligned} x \in a^{-1}(U_i) & \quad\Leftrightarrow \\ a(x) \in \bigcap_{j \in I \setminus \{i\}} k \setminus \{\alpha_j\} & \quad\Leftrightarrow \\ \forall j \in I \setminus \{i\}. a(x) \neq \alpha_j & \quad\Leftrightarrow \\ a(x) = \alpha_i & \quad\Leftrightarrow \\ x \in a^{-1}(\{\alpha_i\}) = G_i. & \end{aligned}$$

This finishes the proof that the G_i are all clopen.

(ii) For each $i \in I$, we have that for all $x \in S_i$, $\left(\sum_{i \in I} \alpha_i \chi_{S_i}\right)(x) = \alpha_i$. Therefore

$$\left(\sum_{j \in J} \beta_j \chi_{T_j}\right)(x) = \alpha_i,$$

so there is some j such that $\alpha_i = \beta_j$. Since the β_j are all distinct, there is only one such j. We define $s(i) = j$, and the map is injective. Arguing in the same manner, but starting with β_j instead of α_i, we see that the map is surjective as well. Since S_i is exactly the set where the function takes the value α_i, we have $S_i = T_{s(i)}$. □

Corollary 5.2.18. *For $k = \mathbb{R}, \mathbb{C}$, we have that* $\mathrm{Simp}(X, k)$ *is the span of* $\mathrm{Proj}(C(X))$.

Proof. By Lemma 5.2.17 every simple function a can be expressed as

$$a = \sum_{i \in I} \alpha_i \chi_{G_i},$$

where I is finite, $\alpha_i \in k$ and G_i are clopen. By Lemma 5.2.13, this expression is

$$a = \sum_{i \in I} \alpha_i c(G_i)$$

and shows that a is a k-linear combination of projections, so $\mathrm{Simp}(X,k) \subseteq \mathrm{span}(\mathrm{Proj}(C(X)))$.

On the other hand, every projection is a $\{0,1\}$-valued, hence simple, and so a k-linear combination of projections is contained in $\mathrm{Simp}(X,k)$ as it is a k-linear subspace of $C(X,k)$. Therefore $\mathrm{span}(\mathrm{Proj}(C(X))) \subseteq \mathrm{Simp}(X,k)$, so we can deduce that $\mathrm{Simp}(X,k) = \mathrm{span}(\mathrm{Proj}(C(X)))$. □

Proposition 5.2.19. $\mathrm{Simp}(X,k)$ *is dense in* $C(X,k)$.

Proof. We use the Stone-Weierstrass theorem [24, Theorem V.8.1]. To do this, we must show that $\mathrm{Simp}(X,k)$ is a unital *-subalgebra of $C(X,k)$ that separates the points.

- $\mathrm{Simp}(X,k)$ is a unital *-subalgebra of $C(X,k)$:

  If $a,b \in \mathrm{Simp}(X,k)$ have ranges $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$, then the range of $a+b$ is a subset of $(\alpha_i+\beta_j)_{(i,j)\in I\times J}$, and the range of ab is a subset of $(\alpha_i\beta_j)_{(i,j)\in I\times J}$, both are finite, and so simple functions are closed under sum and product. Constant functions are continuous and have finitely many values, so are in $\mathrm{Simp}(X,k)$, and this includes the unit. This also means that closure under product implies that for any simple function a and $\alpha \in k$ we have $\alpha \cdot a \in \mathrm{Simp}(X,k)$. This shows that $\mathrm{Simp}(X,k)$ is a unital subalgebra, and suffices in the case $k=\mathbb{R}$. In the case $k=\mathbb{C}$, we have that if $(\alpha_i)_{i\in I}$ is the range of a simple function a, $(\overline{\alpha_i})_{i\in I}$ is the range of a^*, and so a^* is also a simple function.

- $\mathrm{Simp}(X,k)$ separates the points of X:

  Let $x,y \in X$ be such that $x \neq y$ (if this is not possible, we are already finished). Since X is Hausdorff, there are disjoint open sets U,V such that $x \in U$ and $y \in V$. Since X is Stone, U is a union of clopen sets, so there must be a clopen set G such that $x \in G$ and $G \subseteq U$. Since $U \cap V = \emptyset$, we have that $y \notin G$. By Lemma 5.2.13, χ_G for any clopen set G is continuous, and since it takes at most 2 values, it is a simple function. We have $\chi_G(x) = 1$ and $\chi_G(y) = 0$ so the points are separated. □

Lemma 5.2.20. *The space* $\mathrm{Simp}(X,\mathbb{R})$ *is an order-unit space, with the restriction of the positive cone and unit of* $C(X,\mathbb{R})$.

Proof. As it is a linear subspace of $C(X,\mathbb{R})$ it is a vector space. The set $\mathrm{Simp}(X,\mathbb{R})_+ = \mathrm{Simp}(X,\mathbb{R}) \cap C(X,[0,\infty))$ is a cone because the closure under addition and multiplication by positive reals holds for both $\mathrm{Simp}(X,\mathbb{R})$ and $C(X,[0,\infty))$. If $\sum_{i\in I}\alpha_i\chi_{G_i}$ is a simple function, we can define

$$I_+ = \{i \in I \mid \alpha_i \geq 0\} \qquad\qquad I_- = \{i \in I \mid \alpha_i < 0\}$$

and we have

$$\sum_{i\in I}\alpha_i\chi_{G_i} = \sum_{i\in I_+}\alpha_i\chi_{G_i} - \sum_{i\in I_-}-\alpha_i\chi_{G_i},$$

so the positive cone generates $\mathrm{Simp}(X,\mathbb{R})$. If $a \in \mathrm{Simp}(X,\mathbb{R})_+$, then since 1 is an order unit for $C(X,\mathbb{R})$, there is some λ such that $a \leq \lambda \cdot 1$, so 1 is an order unit for $\mathrm{Simp}(X,\mathbb{R})$. It is archimedean because if $a \in \mathrm{Simp}(X,\mathbb{R})$ such that $n \cdot a \leq 1$ for all $n \in \mathbb{N}$, then $a \in C(X,(-\infty,0])$ and so $a \leq 0$ in $\mathrm{Simp}(X,\mathbb{R})$ as well. □

Lemma 5.2.21. *Let A be a Boolean algebra and E a partially ordered vector space with strong unit (such as an order-unit space). If $\mu \in \mathbf{EA}(A,[0,1]_E)$ then it extends to a positive unital map $\mathrm{Simp}(X,\mathbb{R}) \to E$, where $X = \mathrm{Spec}(A)$. This extension operation is a natural isomorphism $\theta_{A,E} : \mathbf{EA}(A,[0,1]_E) \cong \mathbf{poVectu}(\mathrm{Simp}(X,\mathbb{R}),E)$.*

Proof. Given μ, we define $g : \mathrm{Simp}(X,\mathbb{R}) \to E$ as follows:

$$g\left(\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)}\right) = \sum_{i\in I} \alpha_i \mu(a_i)$$

Since a simple function may be expressed as a linear combination of characteristic function in many ways, we prove this is well-defined. We do this by proving that for any sum $\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)}$ can be expressed in reduced form $\sum_{j\in J} \beta_j \chi_{\epsilon_A(b_i)}$ such that

$$\sum_{i\in I} \alpha_i \mu(a_i) = \sum_{j\in J} \alpha_j \mu(b_j),$$

i.e. that the transformation to reduced form still produces the same function $g : C(X,\mathbb{R}) \to E$. We use induction on $|I|$.

- Base cases $|I| = 0$ and $|I| = 1$:

  In these cases the sum is already in reduced form and so there is nothing to prove.

- Inductive step:

  Let $k \in I$ be a distinguished element of I. By the inductive hypothesis, we have

  $$\sum_{i\in I\setminus\{k\}} \alpha_i \chi_{\epsilon_A(a_i)} = \sum_{j\in J} \beta_j \chi_{\epsilon_A(b_j)} \text{ and } \sum_{i\in I\setminus\{k\}} \alpha_i \mu(a_i) = \sum_{j\in J} \beta_j \mu(b_j),$$

  where the β_j are distinct and the b_j disjoint.

  For ease of notation, we define $H_j = \epsilon_A(b_j)$ and $G_i = \epsilon_A(a_i)$. We have

  $$\sum_{i\in I} \alpha_i \chi_{G_i} = \sum_{j\in J} \beta_j \chi_{H_j} + \alpha_k \chi_{G_k},$$

and want to put this into reduced form. We make the definitions

$$
\begin{aligned}
B &= \{j \in J | H_j \cap G_k = \emptyset\} \\
C &= \{j \in J | H_j \cap G_k \neq \emptyset\} \\
D &= \{j \in J | H_j \cap G_k \neq \emptyset \text{ and } H_j \setminus G_k \neq \emptyset\} \\
E &= \begin{cases} B + C + D + \{k\} & \text{if } G_k \setminus \bigcup_{j \in B} H_j \neq \emptyset \\ B + C + D & \text{otherwise} \end{cases}
\end{aligned}
$$

As a first step to getting into reduced form, we make the sets disjoint, using the following definitions

$$
\begin{aligned}
&\beta' : E \to \mathbb{R} && b' : E \to A \\
&\beta'_j = \begin{cases} \beta_j & \text{if } j \in B \\ \alpha_k + \beta_j & \text{if } j \in C \\ \beta_j & \text{if } j \in D \end{cases} && b'_j = \begin{cases} b_j & \text{if } j \in B \\ a_k \wedge b_j & \text{if } j \in C \\ b_j \setminus a_k & \text{if } j \in D \end{cases} \\
&\beta'_k = \alpha_j && b'_k = a_k \setminus \bigvee_{j \in C} b_j
\end{aligned}
$$

In the preceding definitions, we understand expressions $j \in C$, $j \in D$ to be as elements of the disjoint union E, as $D \subseteq C$. We also treat the case of k to be unused if it happens that $k \notin E$, when b'_k would evaluate to 0. We have that b'_j are disjoint for distinct (in E) elements j and also for k. We can define a relation $\sim$ on E as $j \sim j'$ iff $\beta'_j = \beta'_{j'}$ and define

$$
\begin{aligned}
J' &= E/\sim \\
\beta''_h &= \beta'_j && \text{where } h \in J' \text{ and } j \in h \\
b''_h &= \bigvee_{j \in h} b'_j.
\end{aligned}
$$

We now have by definition that the β''_h are distinct and the b''_h are disjoint. Therefore $\sum_{h \in J'} \beta''_h \chi_{\epsilon_A(b''_h)}$ is in reduced form, but we have not yet shown it is equal to $\sum_{i \in I} \alpha_i \chi_{G_i}$. This is proved as follows:

$$
\begin{aligned}
\sum_{h \in J'} \beta''_h \chi_{\epsilon_A(b''_h)} &= \sum_{h \in J'} \beta''_h \sum_{j \in h} \chi_{\epsilon_A(b'_j)} && \text{since } b'_j \text{ are disjoint} \\
&= \sum_{h \in J'} \sum_{j \in h} \beta''_h \chi_{\epsilon_A(b'_j)} \\
&= \sum_{j \in E} \beta'_j \chi_{\epsilon_A(b'_j)} && \text{definition of } \beta'' \\
&= \sum_{j \in B} \beta_j \chi_{H_j} + \sum_{j \in C} (\alpha_k + \beta_j) \chi_{H_j \cap G_k} \\
&\quad + \sum_{j \in D} \beta_j \chi_{H_j \setminus G_k} + \alpha_k \chi_{G_k \setminus \bigcup_{j \in C} H_j}
\end{aligned}
$$

where the last term is 0 if not present in the previous line

$$
\begin{aligned}
&= \sum_{j\in B} \beta_j \chi_{H_j} \\
&+ \sum_{j\in C} \beta_j (\chi_{H_j\cap G_k} + \chi_{H_j\setminus G_k}) \\
&+ \alpha_k \left(\sum_{j\in C} \chi_{H_j\cap G_k} + \chi_{G_k\setminus \bigcup_{j\in C} H_j} \right)
\end{aligned}
$$

because all terms in $C \setminus D$ are zero

$$
\begin{aligned}
&= \sum_{j\in B} \beta_j \chi_{H_j} + \sum_{j\in C} \beta_j \chi_{H_j} + \alpha_k \chi_{G_k} \\
&= \sum_{j\in J} \beta_j \chi_{H_j} + \alpha_k \chi_{G_k} \\
&= \sum_{i\in I\setminus\{k\}} \alpha_i \chi_{G_i} + \alpha_k \chi_{G_k} && \text{inductive hypothesis} \\
&= \sum_{i\in I} \alpha_i \chi_{G_i} \\
&= \sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)}
\end{aligned}
$$

The proof that $\sum_{i\in I} \alpha_i \mu(a_i) = \sum_{h\in J'} \beta''_h \mu(b''_h)$ is similar, using the disjointness of the sets and that μ is an effect algebra morphism at the appropriate points.

We can now prove that this map is a positive unital map.

- The additive part of linearity:

$$
\begin{aligned}
g(a+b) &= g\left(\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)} + \sum_{j\in J} \beta_j \chi_{\epsilon_A(b_j)} \right) = \sum_{i\in I} \alpha_i \mu(a_i) + \sum_{j\in J} \beta_j \mu(b_j) \\
&= g(a) + g(b).
\end{aligned}
$$

- Multiplicative part of linearity:

$$
\begin{aligned}
g(\beta a) &= g\left(\beta \sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)} \right) = g\left(\sum_{i\in I} \beta \alpha_i \chi_{\epsilon_A(a_i)} \right) = \sum_{i\in I} \beta \alpha_i \mu(a_i) \\
&= \beta \sum_{i\in I} \alpha_i \mu(a_i) = \beta g(a).
\end{aligned}
$$

- Preservation of unit:

  We have that 1 is a simple function, and is $\chi_{\epsilon_A(1)}$, so $g(1) = g(\chi_{\epsilon_A(1)}) = \mu(1) = 1$.

- Preservation of positive cone:

  For a to be positive, it is equivalent to have that the coefficients in its eqxpression as a linear combination of characteristic functions are all non-negative. So let $a = \sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)}$ with $\alpha_i \geq 0$ for all $i \in I$. Now, $g(a) = \sum_{i\in I} \alpha_i \mu(a_i)$. Each $\mu(a_i) \in [0,1]_E$ and so is positive, and the fact that positive elements form a cone shows that $g(a)$ is positive.

We define $\theta_{A,E}$ for a Boolean algebra A and a partially ordered vector space with strong unit E to be the operation taking μ to g as defined above. We define an operation $\iota_{A,E} : \mathbf{poVectu}(\mathrm{Simp}(X,\mathbb{R}),E) \to \mathbf{EA}(A,[0,1]_E)$ as

$$\iota_{A,E}(f) = f \circ c_{\mathrm{Spec}A} \circ \epsilon_A.$$

If we show that $\iota_{A,E}$ is the inverse to $\theta_{A,E}$ it follows that it has the correct type. We show that $\theta_{A,E}$ and $\iota_{A,E}$ are mutually inverse as follows. Let $g \in \mathbf{poVectu}(\mathrm{Simp}(\mathrm{Spec}A,\mathbb{R}),E)$ and $\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a)}$ be a simple function on Spec$A$. We see that

$$\begin{aligned}\theta(\iota(g))\left(\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a)}\right) &= \theta(g \circ c \circ \epsilon_A)\left(\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a)}\right) = \sum_{i\in I} \alpha_i (g \circ c \circ \epsilon_A)(a_i)\\ &= \sum_{i\in I} \alpha_i g(\chi_{\epsilon_A(a_i)}) = g\left(\sum_{i\in I} \alpha_i \chi_{\epsilon_A(a_i)}\right).\end{aligned}$$

By extensionality, we have that $\theta \circ \iota = \mathrm{id}$. Now, if we take $f \in \mathbf{EA}(A,[0,1]_E)$ and $a \in A$, we have

$$\iota(\theta(f))(a) = (\theta(f) \circ c \circ \epsilon_A)(a) = \theta(f)(c(\epsilon_A(a))) = \theta(f)(\chi_{\epsilon_A(a)}) = f(a).$$

By extensionality, we have $\iota \circ \theta = \mathrm{id}$, so the two functions are mutually inverse. We now show that the maps $\iota_{A,E}$ form a natural transformation, *i.e.* that

$$\begin{array}{ccc}\mathbf{poVectu}(\mathrm{Simp}(\mathrm{Spec}A,\mathbb{R}),E) & \xrightarrow{\iota_{A,E}} & \mathbf{EA}(A,[0,1]_E)\\ {\scriptstyle \mathbf{BOUS}(\mathrm{Simp}(\mathrm{Spec}(f),\mathbb{R}),g)}\downarrow & & \downarrow{\scriptstyle \mathbf{EA}(f,[0,1]_g)}\\ \mathbf{poVectu}(\mathrm{Simp}(\mathrm{Spec}B,\mathbb{R}),F) & \xrightarrow[\iota_{B,F}]{} & \mathbf{EA}(B,[0,1]_F)\end{array}$$

commutes, where $f \in \mathbf{BA}(B,A)$ and $g \in \mathbf{poVectu}(E,F)$, and hence $\theta_{A,E}$ is a natural transformation also.

Let $h \in \mathbf{BOUS}(\mathrm{Simp}(\mathrm{Spec}(A),\mathbb{R}),E)$, and $b \in B$ and observe that the top right path reduces to

$$\begin{aligned}\mathbf{EA}(f,[0,1]_g)(\iota_{A,E}(h))(b) &= ([0,1]_g \circ \iota_{A,E}(h) \circ f)(b) = [0,1]_g(\iota_{A,E}(h)(f(b)))\\ &= g((h \circ c_{\mathrm{Spec}A} \circ \epsilon_A)(f(b)))\\ &= g(h(c_{\mathrm{Spec}A}(\epsilon_A(f(b))))).\end{aligned}$$

The bottom left path reduces to

$$\begin{aligned}
&\iota_{B,F}(\mathbf{poVectu}(\mathrm{Simp}(\mathrm{Spec}(f),\mathbb{R}),g)(h))(b)\\
&=\iota_{B,F}(g\circ h\circ \mathrm{Simp}(\mathrm{Spec}(f),\mathbb{R}))(b)\\
&=(g\circ h\circ \mathrm{Simp}(\mathrm{Spec}(f),\mathbb{R})\circ c_{\mathrm{Spec}B}\circ \epsilon_B)(b)\\
&=g(h(\mathrm{Simp}(\mathrm{Spec}(f),\mathbb{R})(c_{\mathrm{Spec}B}(\epsilon_B(b)))))\\
&=g(h(\mathrm{Proj}(C(\mathrm{Spec}(f)))(c_{\mathrm{Spec}B}(\epsilon_B(b))))) && \text{Definition of Simp}\\
&=g(h(c_{\mathrm{Spec}A}(\mathrm{Clopen}(\mathrm{Spec}(f))(\epsilon_B(b))))) && \text{Lemma 5.2.15 (iii)}\\
&=g(h(c_{\mathrm{Spec}A}(\epsilon_A(f(b))))) && \text{Naturality of } \epsilon.
\end{aligned}$$

By extensionality, these are equal. □

We write $\mathrm{Simp}(X,[0,1])$ for the unit interval of $\mathrm{Simp}(X,\mathbb{R})$, which is an effect module. By [83, §VII.4], $[0,1]\otimes A$ is the free effect module on A, and so this is isomorphic to $\mathrm{Simp}(X,[0,1])$ because they both satisfy the same universal property. See also [65, Remark 4.13] where the tensor of $[0,1]$ with σ-complete effect algebras is discussed.

Lemma 5.2.22.

(i) If $a\in\mathrm{Simp}(X,k)$ is positive, it has a positive square root in $\mathrm{Simp}(X,k)$.

(ii) $\mathrm{Simp}(X,\mathbb{R})_+$ is dense in $C(X,\mathbb{R})_+$.

Proof.

(i) By Lemma 5.2.17 we can express $a\in\mathrm{Simp}(X,k)$ in reduced form. Since positive elements in $C(X,\mathbb{R})$ are those taking only nonnegative values, we must have $a=\sum_{i\in I}\alpha_i\chi_{\epsilon_A(a_i)}$ with $\alpha_i\geq 0$ in the expression in reduced form. We show that $\sum_{i\in I}\sqrt{\alpha_i}\chi_{\epsilon_A(a_i)}$, which is positive, is a square root of a. We see that

$$\begin{aligned}
\left(\sum_{i\in I}\sqrt{\alpha_i}\chi_{\epsilon_A(a_i)}\right)\left(\sum_{j\in I}\sqrt{\alpha_j}\chi_{\epsilon_A(a_j)}\right) &= \sum_{(i,j)\in I\times J}\sqrt{\alpha_i}\sqrt{\alpha_j}\chi_{\epsilon_A(a_i)}\chi_{\epsilon_A(a_j)}\\
&=\sum_{(i,j)\in I\times J}\sqrt{\alpha_i}\sqrt{\alpha_j}\chi_{\epsilon_A(a_i)\cap\epsilon_A(a_j)}\\
&=\sum_{i\in I}\alpha_i\chi_{\epsilon_A(a_i)},
\end{aligned}$$

which is a. The last step follows from the requirement that the a_i be disjoint in reduced form.

(ii) Let $a\in C(X,\mathbb{R})_+$. By Proposition 5.2.19 there is a sequence $a_i\in\mathrm{Simp}(X,\mathbb{R})$ such that $\lim_{i\to\infty}a_i=a$. By continuity of multiplication $a_i^2\to a^2$, and continuity of square root for positive elements gives $\sqrt{a_i^2}\to\sqrt{a^2}=a$. The sequence $\sqrt{(a_i)^2}$ is positive and in $\mathrm{Simp}(X,\mathbb{R})$, which finishes the proof of density. □

Theorem 5.2.23. *Let A be a Boolean algebra and $X = \mathrm{Spec}(A)$, and E a Banach effect module. Then $\mathbf{BOUS}(C(X,\mathbb{R}),E) \cong \mathbf{EA}(A,[0,1]_E)$, the isomorphism being natural.*

Proof. We define $\theta_{A,E} : \mathbf{EA}(A,[0,1]_E) \to \mathbf{BOUS}(C(X,\mathbb{R}),E)$ as follows. If we take $\theta_{A,E}$ as defined in Lemma 5.2.21, and $\mu \in \mathbf{EA}(A,[0,1]_E)$, then $\theta_{A,E}(\mu)$ is only defined from $\mathrm{Simp}(X,\mathbb{R}) \to E$. However, as g is a map of order-unit spaces it is bounded (Proposition 1.2.8), so it extends to a bounded linear map $C(X,\mathbb{R}) \to E$ by the BLT theorem [109, Theorem I.7]. This extension is unital because it extends g. To show it is positive, let $a \in C(X,\mathbb{R})_+$ and $a_i \in \mathrm{Simp}(X,\mathbb{R})_+$ be a sequence converging to a (using Lemma 5.2.22). Then $f(a_i) = g(a_i) \in E_+$, and so

$$f(a) = f\left(\lim_{i\to\infty} a_i\right) = \lim_{i\to\infty} f(a_i) = \lim_{i\to\infty} g(a_i),$$

and this is positive because E_+ is closed [6, Proposition 1.14]. In summary, $\theta_{A,E}$ is defined by extending the original $\theta_{A,E}$ from simple functions to all continuous functions.

Just as in Lemma 5.2.21, we define a map $\iota_{A,E} : \mathbf{BOUS}(C(X,\mathbb{R}),E) \to \mathbf{EA}(A,[0,1]_E)$ as

$$\iota_{A,E}(f) = f \circ c_X \circ \epsilon_A.$$

The proof that $\iota_{A,E} \circ \theta_{A,E} = \mathrm{id}$ from Lemma 5.2.21 needs no alteration, as $\iota_{A,E}$ only evaluates the extension on simple functions. The proof that $\theta_{A,E} \circ \iota_{A,E} = \mathrm{id}$ can be obtained from that in Lemma 5.2.21 as follows. If $g \in \mathbf{BOUS}(C(X,\mathbb{R}),E)$, and $a \in \mathrm{Simp}(X,\mathbb{R})$, we have already seen that

$$\theta(\iota(g))(a) = g(a).$$

Since g and $\theta(\iota(g))$ are maps of order-unit spaces, they are continuous, and since simple functions are dense (Proposition 5.2.19), we have $\theta(\iota(g)) = g$.

We now only need to show that ι is a natural transformation. The proof of this is in fact identical to the proof that ι is natural in Lemma 5.2.21, because $C(\text{-},\mathbb{R})$ and $\mathrm{Simp}(\text{-},\mathbb{R})$ have the same definition on maps. □

In the case that a POVM takes its values in a C*-algebra, we can extend this to a morphism between C*-algebras.

Corollary 5.2.24. *Let A be a Boolean algebra, $X = \mathrm{Spec}(A)$, and B a C*-algebra. Then*

$$\mathbf{C^*Alg}_{\mathrm{PU}}(C(X),B) \cong \mathbf{EA}(A,[0,1]_B)$$

by a natural isomorphism.

Proof. Given a positive unital map $f : C(X) \to B$, we can define $\iota_{A,B} = \iota_{A,\mathrm{SA}(B)}$, *i.e.*

$$\iota_{A,B}(f) = f \circ c_X \circ \epsilon_A$$

This is an effect algebra morphism $A \to [0,1]_B$ (see Theorem 5.2.23). We also have that $\theta_{A,\mathrm{SA}(B)}$ takes an **EA** morphism $\mu : A \to [0,1]_B$ to a **BOUS** morphism $\mathrm{SA}(C(X)) = C(X,\mathbb{R}) \to \mathrm{SA}(B)$. We know from Proposition 1.2.10 that restriction to the self-adjoint part defines an isomorphism $\mathbf{C^*Alg}_{\mathrm{PU}}(C(X),B) \to \mathbf{BOUS}(C(X,\mathbb{R}),\mathrm{SA}(B))$, so the inverse composed with $\theta_{A,\mathrm{SA}(B)}$, which we shall call $\theta_{A,B}$, is the inverse to $\iota_{A,B}$. The proof that $\iota_{A,B}$ is natural is the same as in Theorem 5.2.23. □

In fact, as positivity and complete positivity are the same if the domain or codomain of a map is a commutative C*-algebra ([128, Chapter IV Corollary 3.5 and Proposition 3.9], so we also have $\mathbf{C^*Alg}_{\mathrm{CPU}}(C(X),B) \cong \mathbf{EA}(A,[0,1]_B)$.

5.2.5 Stonean Spaces

A *stonean* space is a compact Hausdorff space that is *extremally disconnected*. Extremally disconnected means that the closure of any open set is open (and therefore clopen).

We prove the following directly, though it is a consequence of some results we shall cite later.

Lemma 5.2.25. *Every stonean space is a Stone space.*

Proof. Let X be a stonean space. Since it is already compact Hausdorff, we only need to show that it is zero-dimensional, *i.e.* that clopen sets form a basis. So let $x \in X$ be an arbitrary point, and $U \ni x$ an open neighbourhood of x. We want to show that U contains a clopen containing x. Since compact Hausdorff spaces are completely regular, we have that there is a continuous, $[0,1]$-valued function f that is 0 on x and 1 on $X \setminus U$. If we define $V = f^{-1}([0,\frac{1}{2}))$, then V is an open set, so $\mathrm{cl}\,(V)$ is a clopen set. We have

$$\begin{aligned} f(V) &= \left[0,\frac{1}{2}\right) \\ f(\mathrm{cl}\,(V)) &\subseteq \mathrm{cl}\left(\left[0,\frac{1}{2}\right)\right) = \left[0,\frac{1}{2}\right] && \text{by continuity of } f \\ \mathrm{cl}\,(V) &\subseteq f^{-1}\left(\left[0,\frac{1}{2}\right]\right) \subseteq f^{-1}([0,1)) \subseteq U. \end{aligned}$$

So $\mathrm{cl}\,(V)$ is a clopen set in U. Since $x \in V$, we have that $\mathrm{cl}\,(V)$ contains x. □

Lemma 5.2.26. *Let $(G_i)_{i\in I}$ be a family of clopen subsets of a stonean space.*

(i) Then $(G_i)_{i\in I}$ has a least upper bound in the poset of clopen sets, and

$$\bigvee_{i\in I} G_i = \mathrm{cl}\left(\bigcup_{i\in I} G_i\right).$$

(ii) By the above, greatest lower bounds exist as well. If I is not empty,

$$\bigwedge_{i\in I} G_i = \operatorname{int}\left(\bigcap_{i\in I} G_i\right)$$

where the left hand side is the meet in the Boolean algebra of clopen sets.

Proof.

(i) Since it is a union of open sets,

$$\bigcup_{i\in I} G_i$$

is open. Therefore its closure is clopen, and since $G_i \subseteq \bigcup_{i\in I} G_i \subseteq \operatorname{cl}\left(\bigcup_{i\in I} G_i\right)$ for all i, it is an upper bound.

To show it is a least upper bound, let U be a clopen upper bound for $(G_i)_{i\in I}$. Since $G_i \subseteq U$ for all $i \in I$, we have that $\bigcup_{i\in I} G_i \subseteq U$. We can then use the fact that U is closed to get

$$\operatorname{cl}\left(\bigcup_{i\in I} G_i\right) \subseteq U.$$

(ii) Since negation is a contravariant isomorphism of a Boolean algebra to itself, it preserves meets, so

$$\begin{aligned}
\bigwedge_{i\in I} G_i &= \neg \bigvee_{i\in I} \neg G_i \\
&= \neg\operatorname{cl}\left(\bigcup_{i\in I} \neg G_i\right) && \text{By part (i)} \\
&= \neg\operatorname{cl}\left(\neg \bigcap_{i\in I} G_i\right) && \text{de Morgan and } I \text{ nonempty} \\
&= \neg\neg\operatorname{int}\left(\bigcap_{i\in I} G_i\right) \\
&= \operatorname{int}\left(\bigcap_{i\in I} G_i\right).
\end{aligned}$$

□

The preceding two lemmas show that for a stonean space Clopen(X) is a complete Boolean algebra, and $\eta_X : X \to \operatorname{Spec}(\operatorname{Clopen}(X))$ is an isomorphism (using Stone duality, Theorem 5.2.11). It is also the case that the spectrum of any complete Boolean algebra is stonean [50, §21, Theorem 10]. As well as spectra of complete Boolean algebras, stonean spaces arise as spectra of a certain kind of C*-algebra. An AW*-algebra is defined in [13, §1.4, Definition 2], but as we do not need the definition directly, we do not restate it here.

Proposition 5.2.27. *A commutative C*-algebra is an AW*-algebra iff its spectrum is stonean. In particular, the spectrum of a W*-algebra is stonean.*

Proof. We have that for a compact Hausdorff space X, $C(X)$ is an AW*-algebra iff X is stonean [13, §I.7 Theorem 1]. Since the property of being an AW*-algebra is preserved by isomorphism, by Gelfand duality, a C*-algebra A is an AW*-algebra iff Spec(A) is stonean. Since every W*-algebra is an AW*-algebra, the spectrum of any W*-algebra is stonean. □

A *normal state* ϕ on a C*-algebra is one such that for each orthogonal family $(p_i)_{i\in I}$ of projections in A

$$\phi\left(\sum_{i\in I} p_i\right) = \sum_{i\in I}\phi(p_i). \tag{5.1}$$

We first show that this agrees with the other possible definitions of normal.

Proposition 5.2.28. *For a W*-algebra A, the following three conditions on a state $\phi : A \to \mathbb{C}$ are equivalent.*

(i) ϕ *is normal.*

(ii) For all bounded directed sets $(a_i)_{i\in I}$ *in* A

$$\phi\left(\sup_{i\in I} a_i\right) = \sup_{i\in I}\phi(a_i).$$

(iii) ϕ *is* $\sigma(A, A_*)$*-continuous.*

Proof.

- (i) ⇔ (iii): See [128, Corollary III.3.11].
- (ii) ⇔ (iii): We have, by [30, §I.4.2, Theorem 1] that this holds for linear functionals on von Neumann algebras. So we use that every W*-algebra A has a faithful representation $\rho : A \to B(\mathcal{H})$ such that $\rho(A)$ is a von Neumann algebra and ρ, considered as a function $A \to \rho(A)$, is a homeomorphism of the $\sigma(A, A_*)$ topology onto the ultraweak topology ([128, Theorem III.3.5]). This isomorphism also preserves the order structure, and by Lemma 5.2.3(i) preserves least upper bounds. We therefore can deduce the equivalence of (ii) and (iii) from their equivalence for von Neumann algebras. □

The choice of (5.1) as the definition is motivated by the fact that we want normal states to be correctly defined on AW*-algebras, which are not known to always be directed complete. When working on W*-algebras we shall refer to all three conditions equivalently as *normal.*

We can also define what it means for a map to be normal between AW*-algebras. We take a map $f : A \to B$ between AW*-algebras to be *normal* if it preserves the joins of orthogonal families of projections. We need to show this is consistent with the other definitions of normal for W*-algebras.

Proposition 5.2.29. *If $f : A \to B$ is a *-homomorphism between W*-algebras, the following three conditions are equivalent:*

(i) f preserves suprema of orthogonal families of projections.

(ii) f preserves suprema of bounded directed sets.

(iii) f is weak- continuous.*

Proof.

- (iii) ⇒ (i), (ii): If f is weak-* continuous, then if $(p_i)_{i \in I}$ is an orthogonal family of projections in A, (respectively, $(a_i)_{i \in I}$ is a directed family of self-adjoint elements of A), then $(f(p_i))_{i \in I}$ is an orthogonal family of projections in B, by Lemma 5.2.12 (ii) and the fact that f is a *-homomorphism (respectively, $(f(a_i))_{i \in I}$ is a directed family of self-adjoint elements of B). Let $p = \sup_{i \in I} p_i$ (respectively, $a = \sup_{i \in I} a_i$). If ϕ is a weak-* continuous state $B \to \mathbb{C}$, $\phi \circ f$ is weak-* continuous state $A \to \mathbb{C}$. Therefore

$$\begin{aligned} \phi(f(p)) &= (\phi \circ f)(p) \\ &= (\phi \circ f)\left(\bigvee_{i \in I} p_i\right) \\ &= \bigvee_{i \in I} \phi(f(p_i)) && \text{Proposition 5.2.28} \\ &= \phi\left(\bigvee_{i \in I} f(p_i)\right) && \text{Proposition 5.2.28} \end{aligned}$$

(the argument that $\phi(f(a)) = \phi\left(\bigvee_{i \in I} f(a_i)\right)$ is similar, using the other part of Proposition 5.2.28).

Since W*-algebras are separated by their normal states (Lemma 3.6.4), we have that $f(p) = \bigvee_{i \in I} f(p_i)$ (respectively $f(a) = \bigvee_{i \in I} f(a_i)$), as required.

- (i), (ii) ⇒ (iii): Suppose f preserves joins of orthogonal families of projections (respectively, least upper bounds of bounded directed sets of self-adjoint elements). Let ϕ be a weak-* continuous state on B. We know that ϕ preserves joins of orthogonal families of projections, so $\phi \circ f$ does as well, so $\phi \circ f$ is weak-* continuous on A. Let $(a_i)_{i \in I}$ be a weak-* convergent net in A. For all weak-* continuous states ϕ, we have

$$\begin{aligned} \phi\left(\lim_{i \in I} f(a_i)\right) &= \lim_{i \in I} \phi(f(a_i)) && \phi \text{ continuous} \\ &= \lim_{i \in I} (\phi \circ f)(a_i) \\ &= (\phi \circ f)\left(\lim_{i \in I} a_i\right) && \phi \circ f \text{ continuous} \\ &= \phi\left(f\left(\lim_{i \in I} a_i\right)\right). \end{aligned}$$

Since B is separated by its weak-* continuous states (Lemma 3.6.4), we can deduce from the above that $\lim_{i\in I} f(a_i) = f(\lim_{i\in I} a_i)$, *i.e.* f is weak-* continuous. □

We can therefore define the category of AW*-algebras $\mathbf{AW^*Alg}$ to have normal *-homomorphisms as morphisms, and the above result shows that the category of W*-algebras $\mathbf{W^*Alg}$ is a full subcategory. This holds equally well for the categories of commutative AW*-algebras $\mathbf{CAW^*Alg}$, and commutative W*-algebras $\mathbf{CW^*Alg}$.

Lemma 5.2.30. *Let A, B be AW*-algebras and $f : A \to B$ a bijective *-homomorphism. Then f is a normal *-isomorphism,* i.e. *it is normal and has a normal inverse $g : B \to A$.*

Proof. The set-theoretic inverse g is a *-homomorphism by the fact that the axioms for a *-homomorphism are equational, so we can give a proof of linearity, preservation of *, unit and multiplication all along the same lines by replacing each element $b \in B$ appearing in the axiom by $f(g(b))$, doing a rearrangement using the fact that f is a *-homomorphism, and then eliminating $g \circ f$.

When restricted to projections, f and g are isomorphisms of posets and so preserve all joins, in particular, joins of orthogonal families of projections, so are both normal. □

The above implies, in particular, that isomorphisms of W*-algebras are normal according to any of the three equivalent definitions.

Like the characterization of spectra of commutative AW*-algebras in Proposition 5.2.27, there is a characterization of the spectra of commutative W*-algebras. A *hyperstonean* space is a compact Hausdorff space X that is stonean, and such that $C(X)$ is separated by its normal states, *i.e.* if $a, b \in C(X)$ and for all normal states $\phi : C(X) \to \mathbb{C}$ we have $\phi(a) = \phi(b)$, then $a = b$.

Proposition 5.2.31. *A commutative C*-algebra is a W*-algebra iff its spectrum is hyperstonean.*

Proof. This is given in [128, Theorem III.1.18]. □

Just as we say a set has the Baire property if it symmetrically differs from an open set by a nowhere dense set, we can define the sets having the *Claire property* (*cl*open B*aire*) to be those that symmetrically differ from a *clopen* set by a nowhere dense set. Unlike sets with the Baire property, there is no terminological clash in describing such sets as *Claire sets*, so we allow ourselves this. We write $\mathcal{C}\ell(X)$ for the set of Claire sets of a space X.

Corollary 5.2.32. *If X is stonean, for each open set U there is a clopen set G such that $U \triangle G$ is meagre,* i.e. *every open set is Claire.*

Proof. The statement follows from the fact that the stoneanness of X implies $\mathrm{cl}(U)$ is clopen, and $\mathrm{cl}(U) \triangle U = \mathrm{cl}(U) \setminus U$, which is nowhere dense by Lemma 5.2.5 and therefore meagre. □

Corollary 5.2.33. *In a stonean space, for every set S with the Baire property, there exists a clopen set G such that $S \triangle G$ is meagre,* i.e. *the Baire property implies the Claire property.*

Proof. By the assumption that S has the Baire property, there is some open U such that $S \triangle U$ is meagre. Applying the above (Corollary 5.2.32), there is a G such that $U \triangle G$ is meagre. Therefore $S \triangle G$ is meagre, using Proposition 5.2.1(ii). □

Lemma 5.2.34. *For any stonean space X, the map*

$$\mathrm{Clopen}(X) \to \mathcal{C}\ell(X)/\mathcal{M}eagre(X)$$

arising from the inclusion $\mathrm{Clopen}(X) \subseteq \mathcal{C}\ell(X)$ *followed by the quotient modulo meagre sets is an order isomorphism.*

Proof. The map is surjective by definition, so we show first that it is injective. Suppose G_1, G_2 are two clopen sets in the stonean space X such that $G_1 \triangle G_2$ is meagre. Now

$$G_1 \triangle G_2 = (G_1 \cup G_2) \setminus (G_1 \cap G_2) = (G_1 \cup G_2) \cap (X \setminus (G_1 \cap G_2)).$$

This is therefore an open set. Since X is a compact Hausdorff space, by the Baire category theorem [19, IX.5.3 Definition 3 and Theorem 1] every non-empty open subset is not meagre, so $G_1 \triangle G_2$ is empty and therefore $G_1 = G_2$.

Now we show that the order is preserved in each direction.

Suppose $G_1 \subseteq G_2$ are clopen sets. We want to show that $[G_1] \leq [G_2]$. By Lemma 5.2.2, this is so iff $G_1 \setminus G_2$ is meagre. In fact, $G_1 \setminus G_2$ is empty, and therefore meagre.

For the other direction, suppose that G_1, G_2 are clopen and $[G_1] \leq [G_2]$. This implies that $G_1 \setminus G_2$ is meagre. Since the difference of two clopen sets is clopen, $G_1 \setminus G_2$ is a meagre open set, which is therefore empty by the Baire category theorem. Therefore $G_1 \subseteq G_2$. □

We finish by stating a characterization of $C(X)$ of a stonean space X from the literature.

Proposition 5.2.35. *A compact Hausdorff space X is a stonean space iff $C(X)$ is a conditionally complete lattice in the pointwise order, equivalently the order coming from $C(X)$ being a C*-algebra.*

Proof. This is (i) ⇔ (ii) from [128, Proposition III.1.7]. □

Lemma 5.2.36.

(i) A clopen subset of a stonean space is stonean.

(ii) A clopen subset of a hyperstonean space is hyperstonean.

Proof.

(i) Let G be a clopen subset of a stonean space X. Since G is a closed subset of a compact Hausdorff space, it is compact and Hausdorff. Let $U \subseteq G$ be an open subset of G. By Lemma 5.2.6(i) it is an open subset of X. By Lemma 5.2.7 its closure in G equals its closure in X, which is open in X because X is stonean, and is therefore open in G by Lemma 5.2.6(i). Therefore G is stonean.

(ii) Let G be a clopen subset of a hyperstonean space. By (i), G is stonean. If G is empty, we are done because the trivial C*-algebra is a W*-algebra, being the dual space of the 0-dimensional vector space. So we reduce to the case $G \neq \emptyset$. So far, we know that G is a stonean space, and hence $C(G)$ is a conditionally complete lattice (by Proposition 5.2.35), and hence bounded directed-complete, and thus we can define what it is to be a normal state on $C(G)$. We need to show that if $f \in C(G)$ is not zero, there is a normal state ϕ on $C(G)$ such that $\phi(f) \neq 0$. So let f be such a non-zero function, which exists because $G \neq \emptyset$. Using the λ defined in Lemma 5.2.16, we have that $\lambda(f) \in C(X)$, so there is some ψ, a normal state on $C(X)$, such that $\psi(\lambda(f)) \neq 0$. Because λ preserves directed suprema and is positive, $\psi \circ \lambda$ is a normal positive linear functional. We know that $\psi(\lambda(1)) \neq 0$ because for a positive linear functional on a unital C*-algebra, the norm equals the value at 1 [29, Proposition 2.1.4]. We can therefore define

$$\phi = \frac{\psi \circ \lambda}{\|\psi(\lambda(1))\|}$$

which is still normal and positive and is now unital. We have that $\phi(f) \neq 0$ by this definition. Since this works for all f as X is hyperstonean, we have that G is also hyperstonean. □

5.3 Stonean Duality

We now have enough preliminaries to deal with our first duality. Using Gelfand duality, we can get the duality between hyperstonean spaces and commutative W*-algebras, stated by Chris Heunen on MathOverflow [52] (see also [54, §2]). It is based on the duality between complete Boolean algebras and complete homomorphisms, and stonean spaces with open maps proved by Bezhanishvili [14, Corollary 6.10 (2)] using proximity spaces. We give a direct proof here.

Proposition 5.3.1. *Let $f : X \to Y$ be a continuous map between stonean spaces X and Y. The following three statements are equivalent.*

(i) $\mathrm{Clopen}(f) = f^{-1} : \mathrm{Clopen}(Y) \to \mathrm{Clopen}(X)$ *is a map of complete Boolean algebras,* i.e. *preserves joins.*

(ii) f is open.

(iii) For all open sets $U \subseteq Y$, $f^{-1}(\mathrm{cl}\,(U)) = \mathrm{cl}\left(f^{-1}(U)\right)$.

Proof.

- (i) $\Rightarrow$ (ii): Let $G \subseteq X$ be a clopen set. We want to show first that $f(G)$ is open. So define the family $(V_i)_{i\in I}$ to be $\{V \in \text{Clopen}(Y) | f(G) \subseteq V\}$. This family is never empty, because $Y \in (V_i)$. We have therefore that

$$\begin{aligned} \bigwedge_{i\in I} V_i &= \text{int}\left(\bigcap_{i\in I} V_i\right) && \text{Lemma 5.2.26 (ii)} \\ &= \text{int}\left(\text{cl}\left(f(G)\right)\right) && Y \text{ 0-dimensional} \\ &= \text{int}\left(f(G)\right) && f(G) \text{ closed since } G \text{ is compact.} \end{aligned}$$

  Now we have

$$\begin{aligned} f(G) &\subseteq V_i && \text{for all } i \in I \\ G &\subseteq f^{-1}(V_i) && \text{for all } i \in I \\ G &\subseteq \bigwedge_{i\in I} f^{-1}(V_i) \\ G &\subseteq f^{-1}\left(\bigwedge_{i\in I} V_i\right) && f^{-1} \text{ is normal} \\ G &\subseteq f^{-1}(\text{int}\left(f(G)\right)) && \text{statement shown above} \\ f(G) &\subseteq \text{int}\left(f(G)\right). \end{aligned}$$

  Since the interior of a set is always contained in that set, we have $f(G) = \text{int}\left(f(G)\right)$, and so $f(G)$ is open. Thus we have shown that the image of a clopen set is open (and it is also closed, by compactness). Because X is 0-dimensional, every open set U can be expressed as $U = \bigcup_{i\in I} G_i$ for some family $(G_i)_{i\in I}$. Therefore we have that for any open set U

$$f(U) = f\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} f(G_i).$$

  Since $f(U)$ is the union of the open sets $f(G_i)$, it is also open, and so f is an open map.

- (ii) $\Rightarrow$ (iii): As f is an open map, $f(X) \subseteq Y$ is clopen. So without loss of generality we can reduce to the case that f is surjective, using Lemma 5.2.36. Let V be an open subset of Y. Define $U = X \setminus \text{cl}\left(f^{-1}(V)\right)$, which is an open subset of X. We have that $U \subseteq X \setminus f^{-1}(V)$, so $f(U) \cap V = \emptyset$. Then

$$\begin{aligned} V &\subseteq Y \setminus f(U) \\ \text{cl}\left(V\right) &\subseteq Y \setminus f(U) && \text{since } f(U) \text{ is open.} \end{aligned}$$

To prove the implication, we want to show that this inclusion is an equality. So let V' be an open subset of Y such that $f(U) \subseteq V'$ and $V' \cap V = \emptyset$. Such an open set definitely exists, as $f(U)$ satisfies these criteria. Now,

$$\emptyset = f^{-1}(V' \cap V) = f^{-1}(V') \cap f^{-1}(V),$$

so $f^{-1}(V') \subseteq X \setminus f^{-1}(V)$. As it is open, we have $f^{-1}(V') \subseteq X \setminus \mathrm{cl}\left(f^{-1}(V)\right) = U$. Therefore $f(f^{-1}(V')) \subseteq f(U)$. Since f is surjective, $f(f^{-1}(V')) = V'$, so $V' \subseteq f(U)$. Together with the definition of V', we have $V' = f(U)$. Since $\mathrm{cl}\,(V) \subseteq Y \setminus f(U)$, we have that $f(U) \subseteq Y \setminus \mathrm{cl}\,(V)$, and so $Y \setminus \mathrm{cl}\,(V)$ satisfies the criteria for V', thus $f(U) = Y \setminus \mathrm{cl}\,(V)$, and so $\mathrm{cl}\,(V) = Y \setminus f(U)$. We therefore have

$$f^{-1}(\mathrm{cl}\,(V)) = f^{-1}(Y \setminus f(U)) = f^{-1}(Y \setminus f(X \setminus \mathrm{cl}\left(f^{-1}(V)\right))).$$

Let $x \in f^{-1}(\mathrm{cl}\,(V))$, and so $f(x) \in \mathrm{cl}\,(V)$. We then know that $f(x) \notin f(X \setminus \mathrm{cl}\left(f^{-1}(V)\right))$, in other words, that there is no $x' \in X \setminus \mathrm{cl}\left(f^{-1}(V)\right)$ such that $f(x') = f(x)$. Applying de Morgan's law, we have that for all $x' \in X$ such that $f(x') = f(x)$, $x' \notin X \setminus \mathrm{cl}\left(f^{-1}(V)\right)$. We can therefore conclude that $f(x') = f(x)$ implies $x' \in \mathrm{cl}\left(f^{-1}(V)\right)$. In particular, this means that $x \in \mathrm{cl}\left(f^{-1}(V)\right)$. So we have shown that $f^{-1}(\mathrm{cl}\,(V)) \subseteq \mathrm{cl}\left(f^{-1}(V)\right)$.

The inclusion in the other direction holds for any continuous map, but we give the proof here:

$$\begin{aligned} V &\subseteq \mathrm{cl}\,(V) \\ f^{-1}(V) &\subseteq f^{-1}(\mathrm{cl}\,(V)) \\ \mathrm{cl}\left(f^{-1}(V)\right) &\subseteq f^{-1}(\mathrm{cl}\,(V)) && \text{preimage of a closed set is closed.} \end{aligned}$$

- (iii) $\Rightarrow$ (i): Let $(G_i)_{i \in I}$ be a family of clopens in Y. Then

$$\begin{aligned} f^{-1}\left(\bigvee_{i \in I} G_i\right) &= f^{-1}\left(\mathrm{cl}\left(\bigcup_{i \in I} G_i\right)\right) && \text{by Lemma 5.2.26} \\ &= \mathrm{cl}\left(f^{-1}\left(\bigcup_{i \in I} G_i\right)\right) && \text{by assumption} \\ &= \mathrm{cl}\left(\bigcup_{i \in I} f^{-1}(G_i)\right) \\ &= \bigvee_{i \in I} f^{-1}(G_i) && \text{by Lemma 5.2.26. } \square \end{aligned}$$

Proposition 5.3.2. *The following are equivalent for a continuous map $f : X \to Y$, with X, Y stonean.*

(i) f^{-1} is a map of complete Boolean algebras.

(ii) If $N \subseteq Y$ is meagre, $f^{-1}(N)$ is meagre.

(iii) $C(f) : C(Y) \to C(X)$ is normal.

Proof.

- (i) ⇒ (ii): We work up from the definition of a meagre set. Let $N \subseteq Y$ be closed with empty interior. The set $f^{-1}(N) \subseteq X$ is closed by continuity of f. If there is a non-empty $U \subseteq f^{-1}(N)$, then $f(U) \subseteq N$, and $f(U)$ is non-empty. By Proposition 5.3.1 (ii), f is an open map and so $f(U)$ is a non-empty open set in Y, contradicting N having empty interior, so we conclude that no such U can exist and so $f^{-1}(N)$ has empty interior.

  Now let N be nowhere dense, *i.e.* $\mathrm{cl}\,(N)$ has empty interior. We already know that $f^{-1}(\mathrm{cl}\,(N))$ has empty interior. By Proposition 5.3.1 (iii), $\mathrm{cl}\left(f^{-1}(N)\right) = f^{-1}(\mathrm{cl}\,(N))$, so $\mathrm{cl}\left(f^{-1}(N)\right)$ has empty interior and so $f^{-1}(N)$ is nowhere dense.

  Now suppose N is meagre, being equal to $\bigcup_{i=1}^{\infty} N_i$ for a sequence (N_i) of nowhere dense sets. Then

$$f^{-1}(N) = f^{-1}\left(\bigcup_{i=1}^{\infty} N_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(N_i).$$

  The right hand side is a countable union of nowhere dense sets by the previous paragraph, and so is meagre.

- (ii) ⇒ (i): Suppose $N \subseteq Y$ is meagre implies $f^{-1}(N)$ is meagre. To show that f^{-1} is a morphism of complete Boolean algebras, it suffices to show it preserves arbitrary joins, as it already a morphism of Boolean algebras. So let $(G_i)_{i\in I}$ be a family of clopens. We have that

$$f^{-1}\left(\bigvee_{i\in I} G_i \triangle \bigcup_{i\in I} G_i\right) = f^{-1}\left(\bigvee_{i\in I} G_i\right) \triangle \bigcup_{i\in I} f^{-1}(G_i),$$

  and the left hand side is a meagre set by Lemmas 5.2.26 and 5.2.5 and the assumption on f, so the right hand side is too. For the same reason, $\bigvee_{i\in I} f^{-1}(G_i) \triangle \bigcup_{i\in I} f^{-1}(G_i)$ is nowhere dense, hence meagre. By Proposition 5.2.1 (ii), we have that $f^{-1}\left((\bigvee_{i\in I} G_i)\right) \triangle \bigvee_{i\in I} f^{-1}(G_i)$ is meagre, and so as these are two clopen sets, it is a meagre clopen set, and therefore empty by the Baire category theorem, so the two are equal.

- (iii) ⇔ (i): The map $f^{-1} : \mathrm{Clopen}(Y) \to \mathrm{Clopen}(X)$ is exactly $\mathrm{Clopen}(f)$. If $\mathrm{Clopen}(f)$ is a complete Boolean algebra homomorphism, by Lemmas 5.2.15 (iii) and 5.2.3 (iii), $\mathrm{Proj}(C(f)) = c_X \circ \mathrm{Clopen}(f) \circ s_Y$ is a complete Boolean algebra homomorphism. The opposite implication arises from $\mathrm{Clopen}(f) = s_X \circ \mathrm{Proj}(C(f)) \circ c_Y$. □

We define **Stonean** to have Stonean spaces as objects and open (continuous) maps as morphisms. Similarly, we define **HypStonean** to be the full subcategory of **Stonean** on hyperstonean spaces.

The following theorem originates from [52] and [54, §2].

Theorem 5.3.3. *Gelfand duality restricts to give an equivalence* **Stonean** $\simeq$ $\mathbf{CAW^*Alg}^{\mathrm{op}}$, *and* **HypStonean** $\simeq$ $\mathbf{CW^*Alg}^{\mathrm{op}}$.

Proof. The spectrum of a commutative AW*-algebra is stonean by Proposition 5.2.27, and the spectrum of a W*-algebra is hyperstonean by Proposition 5.2.31. All that is needed to show that this restriction to normal and open maps on each side gives an equivalence is to show that normal maps become open maps and vice-versa. This follows by combining Propositions 5.3.1 and 5.3.2 with the original Gelfand duality. □

We have a categorical equivalence we can prove using this duality, but to define one of the categories involved, we first need a notion for Boolean algebras. A *state* on a Boolean algebra A is an element of $\mathbf{EA}(A, [0,1])$, equivalently it is a function $\phi : A \to [0,1]$ such that $\phi(a \vee b) = \phi(a) + \phi(b)$ if $a \wedge b = 0$. One might also call it a finitely-additive probability measure. A *normal state* on a complete Boolean algebra A is a state that is *completely additive*, *i.e.* for any pairwise disjoint family $(a_i)_{i\in I}$ in A, we have $\phi\left(\bigvee_{i\in I} a_i\right) = \sum_{i\in I} \phi(a_i)$.

Lemma 5.3.4. *A state ϕ on a complete Boolean algebra A is normal iff it preserves directed suprema iff it preserves directed infima.*

Proof. See [41, 363R (i), (iii) and 363O], where this is used as an alternative definition of a normal state and is proven equivalent to complete additivity. □

Lemma 5.3.5. *Under the isomorphism $\theta_{A,\mathbb{C}}$ from Corollary 5.2.24, completely additive states on a complete Boolean algebra A correspond to normal states on $C(\mathrm{Spec}(A))$.*

Proof. Suppose ϕ is a completely additive state on A. We have that $\theta_{A,\mathbb{R}}(\phi)$ is a state on $C(\mathrm{Spec}(A))$. To show it is normal, we must show that for any orthogonal family of projections $(p_i)_{i\in I}$, $\theta_{A,\mathbb{C}}(\phi)\left(\bigvee_{i\in I} p_i\right) = \sum_{i\in I} \theta_{A,\mathbb{C}}(\phi)(p_i)$. We have the poset isomorphism (hence complete Boolean isomorphism by Lemma 5.2.3 (iii)) $c_{\mathrm{Spec}A} \circ \epsilon_A : A \to \mathrm{Proj}(C(\mathrm{Spec}(A)))$, so each p_i can be expressed as $c_{\mathrm{Spec}A}(\epsilon_A(a_i))$. Now

$$\begin{aligned}\theta_{A,\mathbb{C}}(\phi)\left(\bigvee_{i\in I} p_i\right) &= \theta_{A,\mathbb{C}}(\phi)\left(\bigvee_{i\in I} c_{\mathrm{Spec}A}(\epsilon_A(a_i))\right)\\ &= \theta_{A,\mathbb{C}}(\phi)\left(c_{\mathrm{Spec}A}\left(\epsilon_A\left(\bigvee_{i\in I} a_i\right)\right)\right) = \iota_{A,\mathbb{C}}(\theta_{A,\mathbb{C}}(\phi))\left(\bigvee_{i\in I} a_i\right)\\ &= \phi\left(\bigvee_{i\in I} a_i\right) = \sum_{i\in I} \phi(a_i) = \sum_{i\in I} \iota_{A,\mathbb{C}}(\theta_{A,\mathbb{C}}(\phi))(a_i)\end{aligned}$$

$$= \sum_{i\in I} \theta_{A,\mathbb{C}}(\phi)(c_{\mathrm{Spec}A}(\epsilon_A(a_i))) = \sum_{i\in I} \theta_{A,\mathbb{C}}(\phi)(p_i).$$

For the other direction, let ϕ be a normal state on $C(\mathrm{Spec}(A))$, and we want to show that $\iota_{A,\mathbb{C}}(\phi)$ is a normal state on A. If $(a_i)_{i\in I}$ is a family of pairwise disjoint elements of A, we can show $\iota_{A,\mathbb{C}}(\phi)\left(\bigvee_{i\in I} a_i\right) = \sum_{i\in I} \iota_{A,\mathbb{C}}(\phi)(a_i)$ in as similar manner to the proof above, so we omit this part. □

We can now define a *measure algebra* to be a complete Boolean algebra A that is separated by its normal states, *i.e.* such that for any element $a \in A$ such that for all normal states ϕ, $\phi(a) = 0$, we have $a = 0$. This notion was introduced by Dmitri Pavlov [100] under the name measurable locale, but we use the name measure algebra to explicitly draw the connection to Fremlin's localizable measure algebras [41, 322A (e)]. The reason for the introduction of this notion will be apparent after the next Proposition. First we need a lemma, however.

Lemma 5.3.6. *Let A be a complete Boolean algebra.*

(i) If $b \in A$ and ψ is a normal state on A such that $\psi(b) \neq 0$, then there exists a normal state ϕ such that $\phi(b) = 1$ and for any $a \in A$ such that $a \wedge b = 0$, $\phi(a) = 0$.

(ii) If A is a measure algebra, and $a \in \mathrm{Simp}(\mathrm{Spec}A, \mathbb{C})$, then there exists a normal state ϕ on $C(\mathrm{Spec}(A))$ such that $|\phi(a)| = \|a\|$.

Proof.

(i) We define ϕ as

$$\phi(a) = \frac{\psi(a \wedge b)}{\psi(b)}.$$

This expression is defined because $\psi(b) \neq 0$. We have

$$\phi(1) = \frac{\psi(1 \wedge b)}{\psi(b)} = 1,$$

so $\phi : A \to [0, 1]$. To show that it is completely additive, let $(a_i)_{i\in I}$ be a pairwise disjoint family in A. Then

$$\phi\left(\bigvee_{i\in I} a_i\right) = \frac{\psi\left(\left(\bigvee_{i\in I} a_i\right) \wedge b\right)}{\psi(b)} = \frac{\psi\left(\bigvee_{i\in I} a_i \wedge b\right)}{\psi(b)} = \sum_{i\in I} \frac{\psi(a_i \wedge b)}{\psi(b)}$$
$$= \sum_{i\in I} \phi(a_i).$$

Now suppose that $a \wedge b = 0$. Then

$$\phi(a) = \frac{\psi(a \wedge b)}{\psi(b)} = \frac{\psi(0)}{\psi(b)} = 0.$$

Finally, we see that

$$\phi(a) = \frac{\psi(a \wedge a)}{\psi(a)} = 1.$$

(ii) Let $a \in \text{Simp}(\text{Spec}A, \mathbb{C})$. If $a = 0$ then this holds for any normal state, so we reduce to the case that $a \neq 0$. Express a in reduced form, using Lemma 5.2.17, as $\sum_{i \in I} \alpha_i \chi_{\epsilon_A(a_i)}$. By definition of the norm on $C(\text{Spec}A)$, we have that $\|a\| = \max_{i \in I} |\alpha_i|$, so pick the i for which this maximum is achieved, calling it k, so that $\|a\| = |\alpha_k|$. Since A is a measure algebra, there is some normal state on A, ψ, such that $\psi(a) \neq 0$. Using the previous part, we can find a normal state ψ' such that $\psi'(a_k) = 1$ and $\psi'(a_i) = 0$ for all other $i \neq k$, as they are disjoint from a_k. We define $\phi = \theta_{A,\mathbb{C}}(\psi')$ (Corollary 5.2.24). This is a normal state on $C(\text{Spec}A)$ by Lemma 5.3.5, and

$$\phi(a) = \theta_{A,\mathbb{C}}(\psi') \left(\sum_{i \in I} \alpha_i \chi_{\epsilon_A(a_i)} \right) = \sum_{i \in I} \alpha_i \psi'(a_i) = \alpha_k.$$

Therefore $|\phi(a)| = |\alpha_k| = \|a\|$, as required. □

Proposition 5.3.7. *A Boolean algebra is a measure algebra iff its spectrum is hyperstonean.*

Proof. We already have that a Boolean algebra is complete iff its spectrum is stonean, so we reduce to showing that a Boolean algebra A is separated by its normal states iff $C(\text{Spec}(A))$ is separated by its normal states.

Suppose $\text{Spec}(A)$ is hyperstonean, *i.e.* $C(\text{Spec}(A))$ is separated by its normal states, and let $a \in A$, $a \neq 0$. We want to show that there is a normal state on A such that $\phi(a) \neq 0$. We have the isomorphism $A \to \text{Proj}(C(\text{Spec}(A)))$ by composing isomorphisms to get $c_{\text{Spec}A} \circ \epsilon_A^{\mathbf{BA}}$. As $\text{Spec}(A)$ is hyperstonean, we know that there is a $\phi : C(\text{Spec}(A))$ such that $\phi(c_{\text{Spec}A}(\epsilon_A^{\mathbf{BA}}(a))) \neq 0$. By Lemma 5.3.5, $\phi \circ c_{\text{Spec}A} \circ \epsilon_A^{\mathbf{BA}}$ is a normal state on A, so we are done.

Now, on the other hand, suppose that A is a measure algebra, and let $a \in C(\text{Spec}(A))$. By Proposition 5.2.19, simple functions are dense in $C(\text{Spec}(A))$, so there is a simple function b such that

$$\|a - b\| < \frac{\|a\|}{3}. \tag{5.2}$$

Using Lemma 5.3.6(ii) we can obtain a normal state ϕ such that $|\phi(b)| = \|b\|$. We will be needing a lower bound for this quantity in terms of $\|a\|$, so we observe that

$$\|a\| = \|b - (b - a)\| \leq \|b\| + \|b - a\| < \|b\| + \frac{\|a\|}{3},$$

using (5.2). Rearranging, we get

$$\|b\| > \frac{2}{3}\|a\|. \tag{5.3}$$

We can now see that

$$|\phi(b)| = |\phi(a) - (\phi(a) - \phi(b))| \leq |\phi(a)| + |\phi(a) - \phi(b)|.$$

So with rearrangement, we get

$$\begin{aligned}
|\phi(a)| &\geq |\phi(b)| - |\phi(a) - \phi(b)| \\
&= \|b\| - |\phi(a-b)| \\
&> \frac{2}{3}\|a\| - |\phi(a-b)| && \text{by (5.3)} \\
&\geq \frac{2}{3}\|a\| - \|a-b\| && \text{Lemma 1.2.3} \\
&> \frac{2}{3}\|a\| - \frac{1}{3}\|a\| && \text{by (5.2)} \\
&= \frac{1}{3}\|a\|.
\end{aligned}$$

If $a \neq 0$, we know $\|a\| \neq 0$, so this shows $\phi(a) \neq 0$, as required. Since a is arbitary, this shows that $C(\mathrm{Spec}(A))$ is separated by its normal states, so $\mathrm{Spec}(A)$ is not only stonean but hyperstonean. ☐

We define **MeasAlg** to be the full subcategory of **CBA** on measure algebras.

Corollary 5.3.8. *The functor* Proj : $\mathbf{CAW^*Alg} \to \mathbf{CBA}$ *is an equivalence, and restricts to an equivalence* $\mathbf{CW^*Alg} \to \mathbf{MeasAlg}$.

Proof. By Theorem 5.3.3 we have that Spec : $\mathbf{CAW^*Alg}^{\mathrm{op}} \to \mathbf{Stonean}$ is an equivalence by restricting Gelfand duality to normal and open maps on each side, and Spec : $\mathbf{CW^*Alg}^{\mathrm{op}} \to \mathbf{HypStonean}$ is also an equivalence. We also have that Clopen : $\mathbf{Stonean} \to \mathbf{CBA}^{\mathrm{op}}$ is an equivalence by restricting Stone duality and using Proposition 5.3.1 to show that open maps correspond to complete Boolean homomorphisms. Using Proposition 5.3.7 we see that if we restrict to **HypStonean** the functor's range is contained in $\mathbf{MeasAlg}^{\mathrm{op}}$ and that Clopen is still essentially surjective when restricted because each measure algebra A is isomorphic to the clopens on a hyperstonean space, namely $\mathrm{Spec}(A)$. We know that in each case, therefore, Clopen ∘ Spec is an equivalence. Since Clopen ∘ Spec is naturally isomorphic to Proj by Corollary 5.2.14 (the naturality following from Lemma 5.2.15 (iii)), we have that Proj is an equivalence in both cases. ☐

5.4 Further Preliminaries

5.4.1 Measure Theory

We can now start to give the measure-theoretic definitions we will need to define the category $\mathcal{M}eas$[3] and the functor L^∞. Our main reference is Fremlin [39, 40, 41, 42], as Fremlin deals extensively with the adjustments necessary to deal with measures that are not σ-finite, whereas σ-finiteness is a standing assumption in many other texts.

We cannot get by with only σ-finite measure spaces for two reasons. To explain them, we will have to make reference to concepts that will be defined

[3] Which is used to define **Meas**.

later. The first is that we would like $L^\infty : \mathcal{M}eas \to \mathbf{CW^*Alg}^{\mathrm{op}}$ to extend the functor $\ell^\infty : \mathbf{Set} \to \mathbf{CW^*Alg}^{\mathrm{op}}$ via the inclusion $\mathbf{Set} \hookrightarrow \mathcal{M}eas$ that maps X to $(X, \mathcal{P}(X), \kappa)$ where κ is the counting measure. The measure κ is σ-finite iff X is countable, so we would have to restrict to countable sets. In fact, there is no σ-finite measure that would give $\ell^\infty(X)$ for X uncountable, as every σ-finite measure on $(X, \mathcal{P}(X))$ has some nullsets that are not equal to 0.

The second reason is that not all commutative W*-algebras occur as L^∞ of some σ-finite measure space. If (X, Σ, μ) is σ-finite, then $\mathrm{Proj}(L^\infty(X, \Sigma, \mu)) \cong \mathrm{BA}(X, \Sigma, \mu)$ is a Boolean algebra with the countable chain condition [41, Theorem 322B (c), Proposition 322G], whereas we can see $\mathcal{P}(X) \cong \mathrm{Proj}(\ell^\infty(X))$ has the countable chain condition iff X is countable, by considering the antichain $(x)_{x \in X}$. With these remarks out of the way, we return to the definitions.

A *measurable space* is a pair (X, Σ) where X is set and $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra. In certain cases where no confusion can result we will use X to refer to the pair (X, Σ). A function $f : X \to Y$, where (X, Σ) and (Y, Θ) are measurable spaces, is said to be *measurable* if for all $S \in \Theta$, $f^{-1}(S) \in \Sigma$. The analogy to continuous maps of topological spaces is clear. Measurable spaces and measurable functions form a category $\mathcal{M}\acute{e}s$, which is the base category of the Giry monad [47], but this will play only a small rôle in the rest of this chapter. A *measure* μ on a measurable space (X, Σ) is a function $\Sigma \to [0, \infty]$ that is countably additive (including $\mu(\emptyset) = 0$). We say it is *finite* if $\mu(S) \neq \infty$ for all $S \in \Sigma$, equivalently if $\mu(X) \neq \infty$. A *signed measure* is a function $\Sigma \to \mathbb{R}$ that is countably additive. If we have a measurable map f as above and a measure μ on (X, Σ), the *image measure* $f_*(\mu)$ on (Y, Θ) is defined as follows:

$$f_*(\mu)(S) = \mu(f^{-1}(S)).$$

Lemma 5.4.1. *The image measure of a measure is a measure, and the image measure of a signed measure is a signed measure.*

Proof. If $S \in \Theta$, we have that $f_*(\mu)(S) = \mu(f^{-1}(S))$ which is an element of $[0, \infty]$ if μ is a measure and $\mathbb{R}$ if μ is a signed measure. So $f_*(\mu)$ is correct as a function on sets. Now let $\{S_i\}_{i \in \mathbb{N}}$ be a sequence of sets in Θ, that are disjoint and possibly all empty.

$$f_*(\mu)\left(\bigcup_{i=1}^{\infty} S_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} S_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(S_i)\right).$$

The map $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a homomorphism of complete Boolean algebras, so it preserves disjointness, and $\{f^{-1}(S_i)\}_{i \in \mathbb{N}}$ is a disjoint sequence. So

$$\mu\left(\bigcup_{i=1}^{\infty} f^{-1}(S_i)\right) = \sum_{i=1}^{\infty} \mu(f^{-1}(S_i)) = \sum_{i=1}^{\infty} f_*(\mu)(S_i).$$

□

Lemma 5.4.2. *The* $-_*$ *operation preserves composition,* i.e. *for measurable maps* $f : (X, \Sigma) \to (Y, \Theta)$ *and* $g : (Y, \Theta) \to (Z, \Omega)$*, we have*

$$g_* \circ f_* = (g \circ f)_*$$

Proof. Let μ be a measure on Σ and $U \in \Omega$. Then

$$\begin{aligned} g_*(f_*(\mu))(U) &= f_*(\mu)(g^{-1}(U)) = \mu(f^{-1}(g^{-1}(U))) \\ &= \mu((g \circ f)^{-1}(U)) = (g \circ f)_*(\mu)(U). \end{aligned}$$

□

In the cases where the measures $\mu(X) = 1$, Lemmas 5.4.1 and 5.4.2 are special cases of the functoriality of the Giry monad.

A *support* for a measure μ is a set $E \in \Sigma$ such that if $F \in \Sigma$ and $F \cap E = \emptyset$, we have that $\mu(F) = 0$. After introducing the notion of support, we find it convenient to prove a lemma about it.

Lemma 5.4.3. *If* $S \in \Sigma$ *is a support for* μ *and* $D \in \Sigma$.

$$\mu(D) = \mu(S \cap D)$$

Proof. We have that $\mu(D) = \mu((S \cap D) \cup (D \setminus S)) = \mu(S \cap D) + \mu(D \setminus S)$. Because $(D \setminus S) \cap S = \emptyset$, $\mu(D \setminus S) = 0$, and the result follows. □

A *measure space* is a triple (X, Σ, μ), where X is a set, $\Sigma \subseteq \mathcal{P}(X)$ a σ-algebra and $\mu : \Sigma \to [0, \infty]$ a (countably additive) measure. When no confusion can result, we will sometimes use X to refer to the triple (X, Σ, μ). A measure space is *complete* if $S \subseteq N$ and $\mu(N) = 0$ implies $S \in \Sigma$ [40, Definition 211A]. If we have a measure space (X, Σ, μ), there exists a *completion* $(X, \hat{\Sigma}, \hat{\mu})$ [40, 212C, 212D]. Now, $\Sigma \subseteq \hat{\Sigma}$ and $\hat{\mu}|_\Sigma = \mu$, and given any element $S \in \hat{\Sigma}$ there exists $S' \in \Sigma$ such that $S \triangle S'$ is a μ-nullset, *i.e.* there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $S \triangle S' \subseteq N$. If (X, Σ, μ) was already complete to start with, then $\hat{\Sigma} = \Sigma$ and $\hat{\mu} = \mu$. Given (X, Σ, μ), we call elements of $\hat{\Sigma}$ the μ-measurable sets.

It is possible to manufacture measure spaces with quite inconvenient properties, even if they are complete. In particular, it is not the case that for every measure space, $L^\infty(X, \Sigma, \mu)$ is a commutative W*-algebra, it might only be a commutative C*-algebra. Localizable measure spaces were introduced in [119] to be exactly the measure spaces where $L^\infty(X, \Sigma, \mu)$ is a W*-algebra, but this notion is not good enough for a duality theorem. We will now give the definitions of some properties measure spaces may have that we shall see (Proposition 5.4.17) always produce W*-algebras. A measure space (X, Σ, μ) is *finite* if μ is finite as a measure. Probability measures are of course always finite. A measure space is σ*-finite* if there is a countable partition $(S_i)_{i \in I}$, each $S_i \in \Sigma$ and $\mu(S_i) < \infty$. The real line $(\mathbb{R}, \Sigma, \mu)$ where Σ is the Borel sets and μ the Lebesgue measure is a well-known example of a σ-finite measure space that is not finite.

One property many measure spaces that occur in practice have is being strictly localizable. A measure space is *strictly localizable* [40, Definition 211E] if it has a *decomposition* $\{S_i\}_{i \in I}$, where $S_i \in \Sigma$ for all $i \in I$, the S_i are pairwise disjoint, and the following three axioms are satisfied:

(i) $\mu(S_i) < \infty$ for all i.

(ii) If $S \cap S_i \in \Sigma$ for all $i \in I$, $S \in \Sigma$.

(iii) For all $S \in \Sigma$, $\mu(S) = \sum_{i \in I} \mu(S \cap S_i)$.

The sum is interpreted as the supremum of the finite sums, and so is allowed to be ∞, and it must match $\mu(S)$ even in these cases.

In the case that $|I| = \aleph_0$, strict localizability reduces to σ-finiteness as the second two axioms are implied by Σ being a σ-algebra and μ being countably additive, respectively. The reasons for using strictly localizable measure spaces are that we shall see that, up to isomorphism, every measure algebra, and therefore every commutative W*-algebra arises from a strictly localizable measure space (Theorem 5.6.9), strict localizability is needed to prove the fullness of the functor L^∞, and that almost all measure spaces occuring in practice, if they are not already σ-finite, are still strictly localizable. For instance $(X, \mathcal{P}(X), \kappa)$ where κ is the counting measure, and the Haar measure on any locally compact group are strictly localizable, but neither is necessarily σ-finite.

Examples of measure spaces that are not strictly localizable are rather artificial. Consider the space (X, Σ, μ) where X is an uncountable set, Σ is the σ-algebra containing the countable and cocountable[4] sets, and μ the counting measure restricted to Σ.

Counterexample 5.4.4. *The space* (X, Σ, μ) *is not strictly localizable.*

Proof. Suppose that $(S_i)_{i \in I}$ is a decomposition of X, without loss of generality discarding any empty sets. As $\mu(S_i) < \infty$ for all $i \in I$, we know that each S_i is finite. Since a countable union of countable sets is countable, I is uncountable. Partition I into two uncountable sets J, J' and define $Y = \bigcup_{j \in J} S_j$ and $Y' = \bigcup_{j \in J'} S_j$, which are two complementary uncountable sets in X, so $Y \notin \Sigma$ and $Y' \notin \Sigma$. But $Y \cap S_i$ is either S_i or $\emptyset$, and in both cases these sets are in Σ, contradicting axiom (ii) for a decomposition. □

In fact, Theorem 5.4.9 shows that (X, Σ, μ) is not even localizable.

As well as strict localizability, another useful property for measure spaces is *compactness*. To define this, we first define a compact class. First, a family of sets $\mathcal{K}' \subseteq \mathcal{P}(X)$ has the *finite intersection property* if for each finite subfamily $(K_i)_{i \in I}$, $K_i \in \mathcal{K}'$, we have $\bigcap_{i \in I} K_i \neq \emptyset$. A family of sets $\mathcal{K} \subseteq \mathcal{P}(X)$ is said to be a *compact class* if for any family $\mathcal{K}' \subseteq \mathcal{K}$ with the finite intersection property, $\bigcap \mathcal{K}' \neq \emptyset$. This is an axiomatization of how closed sets in compact spaces behave. A measure μ is said to be *inner regular* with respect to $\mathcal{K}$ if for all $S \in \Sigma$,

$$\mu(S) = \sup\{\mu(K) \mid K \in \mathcal{K} \cap \Sigma \text{ and } K \subseteq S\}.$$

A measure space (X, Σ, μ) is *compact* if there exists a compact class $\mathcal{K}$ such that μ is inner regular with respect to $\mathcal{K}$.

[4] sets with countable complement

This is actually Fremlin's modified notion of compactness. A compact measure space in the literature [89, 98] usually means what Fremlin calls a countably compact space. Fremlin's notion is the right one for showing the fullness of L^∞, as he already observed [41, 343 Notes and comments].

Compact measure spaces are an abstract version of measure spaces where μ is a Radon measure. Most measure spaces occurring in practice originate from topological spaces and are Radon measures. We summarize why this is so here.

Example 5.4.5. *Any locally finite measure on a Polish space is Radon, and so compact. Any Haar measure on a locally compact group is Radon, and so compact. The counting measure (with the powerset σ-algebra) is compact, with finite sets as a compact class.*

Proof.

- Polish spaces: Let X be a Polish space and $\mu : \mathcal{B}o(X) \to [0, \infty]$ a locally finite measure. If μ is finite, then we can rescale μ to a probability measure and apply [15, Theorem 1.1 and Theorem 1.4] to conclude that μ is inner regular. We therefore reduce to the case that $\mu(X) = \infty$. A Polish space is a separable metric space, and so is second countable [73, p. 3]. It is therefore Lindelöf, *i.e.* every open cover has a countable subcover [75, Chapter 1, Theorem 15]. Since μ is locally finite, each point x has an open neighbourhood U_x such that $\mu(U_x) < \infty$. Since each point is included, the family of sets $(U_x)_{x\in X}$ is an open cover of X, so has a countable subcover $(V_i)_{i\in\mathbb{N}}$. If we define $S_i = V_i \setminus \bigcup_{j=1}^{i-1} V_j$, we have a countable disjoint family of Borel sets of finite measure whose union is X, showing that $(X, \mathcal{B}o(X), \mu)$ is σ-finite. Without loss of generality, if $\mu(S_i) = 0$ for some i, we can amalgamate it with the next set of strictly positive measure, which must exist because $\mu(X) = \infty$, and so we can reduce to the case that $\mu(S_i) > 0$ for all $i \in \mathbb{N}$. We then define $\nu_i = \chi_{S_i} \cdot \mu$. This is a finite measure on a Polish space, so is inner regular with respect to compact sets. Since $\mu(S) = \sum_{i\in\mathbb{N}} \nu_i(S)$, μ is inner regular with respect to compact sets too. This also shows that any σ-finite measure on a standard Borel space is compact, being inner regular with respect to the compact sets of any Polish space defining the Borel structure. Being σ-finite, completions of measure spaces of the above type are objects in $\mathcal{M}eas$.

- Haar measures: The fact that the Haar measure on a locally compact group G is Radon can either be seen by defining it as a positive linear functional on $C_c(G)$ [93, §II.4, Theorem 1] and applying the locally compact version of the Riesz representation theorem [114, Thm. 2.14]. It is also possible to prove this in the usual measure theoretic setting [42, Theorem 441E], and that $(G, \widehat{\mathcal{B}o(G)}, \mu)$ is strictly localizable [42, Theorem 415A], hence an object of $\mathcal{M}eas$.

- Counting measures: The fact that the counting measure on X is inner regular with respect to finite sets is proven by considering the two cases for a subset $S \subseteq X$, when it is finite or infinite. If it is finite, we can take S to be its own approximation by finite sets. If S is infinite, there are finite sets of arbitrarily large size in S, so the measure is also approximated from below by finite sets. The final step is to show that finite sets form a compact class. Suppose $\mathcal{K}' \subseteq \mathcal{P}_{\text{fin}}(X)$ is a family of sets with the finite intersection property, and suppose for a contradiction that $\bigcap \mathcal{K}' = \emptyset$. If $K \in \mathcal{K}'$ is a finite set, it must be the case that for each $x \in K$ there is a $L_x \in \mathcal{K}'$ such that $x \notin K$. But we therefore have a K-indexed, hence finite, family of sets in $\mathcal{K}'$ with empty intersection, contradicting the finite intersection property.

Counting measures are also evidently strictly localizable with decomposition into singletons, so counting measures are objects of $\mathcal{M}eas$. □

To get an example of a measure space that is not compact, take an uncountable set X, take Σ to be the σ-algebra of countable and cocountable sets, and take μ to be the measure such that $\mu(S) = 0$ if S is countable and $\mu(S) = 1$ if S is cocountable.

Counterexample 5.4.6. (X, Σ, μ) *is not a compact measure space.*

Proof. Assume for a contradiction that (X, Σ, μ) is compact. Let $\mathcal{K}$ be a compact class in X, with respect to which μ is inner regular. Define $\mathcal{K}' = \{K \in \mathcal{K} | \mu(K) = 1\}$. Since $\mu(X) = 1$ there must be a $K \in \mathcal{K}$ such that $\mu(K) > 0$, and hence $\mu(K) = 1$ by the 2-valuedness of μ, so $\mathcal{K}'$ is nonempty. In fact, by the preceding argument, every cocountable subset of X contains some $K \in \mathcal{K}$ with $\mu(K) = 1$.

Now, if $J, K \in \mathcal{K}'$, we have that $1 = \mu(J \cup K) = \mu(J \setminus K) + \mu(J \cap K) + \mu(K \setminus J)$. Since $J \setminus K \subseteq X \setminus K$, it has μ-measure 0, as does $K \setminus J$, so $\mu(J \cap K) = 1$. This extends to intersections of finite subfamilies of $\mathcal{K}'$, and so any finite intersection of elements of $\mathcal{K}'$ is nonempty, so by compactness $\bigcap \mathcal{K}'$ is nonempty. However, because $X \setminus \{x\}$ is cocountable, and therefore of μ-measure 1, and so contains an element of $\mathcal{K}'$, we have that for every $x \in X$ there is a $K_x \in \mathcal{K}'$ not containing x. The intersection $\bigcap_{x \in X} K_x$ is therefore empty, so $\bigcap \mathcal{K}'$ must be empty, a contradiction. □

The Radon-Nikodym theorem takes on a slightly different form in spaces that are not σ-finite. To state it, we need the notion of one measure being truly continuous to another.

Definition 5.4.7. *A (finitely additive and finite) signed measure $\alpha : \Sigma \to \mathbb{R}$ is* truly continuous[5] *to a measure $\mu : \Sigma \to [0, \infty]$ if one of the following three equivalent conditions holds:*

(i) For all $\epsilon > 0$, there exist $E \in \Sigma$ and $\delta > 0$ with $\mu(E) < \infty$ such that for all $F \in \Sigma$, $\mu(E \cap F) \leq \delta$ implies $|\alpha(F)| \leq \epsilon$. [40, 232A (b)]

[5] In Fremlin's terminology

*(ii) The measure α is countably additive, α is absolutely continuous to μ (*i.e. *$\mu(E) = 0 \Rightarrow \alpha(E) = 0$), and if $E \in \Sigma$ with $\alpha(E) \neq 0$, there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\alpha(E \cap F) \neq 0$. [40, 232B (b)]*

(iii) The measure α is countably additive and absolutely continuous to μ, and there exists a sequence (E_n) of sets $E_n \in \Sigma$ such that $\mu(E_n) < \infty$ and if $F \in \Sigma$ and $F \cap (\bigcup_n E_n) = \emptyset$ implies $\alpha(F) = 0$ [40, 232X (a)]. In other words, the set $E = \bigcup_{n=1}^{\infty} E_n$ is a support for α, or α has μ-σ-finite support.

Proof. As the last one is an exercise, we offer a proof here that (ii) $\Leftrightarrow$ (iii):

- (ii) $\Rightarrow$ (iii):

  We have that α is absolutely continuous and countably additive by assumption, so we only need to show that it has μ-σ-finite support. We use Zorn's lemma as follows. Consider the poset P of countable (including finite) disjoint families of measurable subsets S such that $|\alpha|(S) > 0$, $\mu(S) < \infty$, ordered by extension.

  We show that there are no uncountable chains in P as follows. Suppose there is an uncountable chain of countable sequences in P. We can take the union of it, which gives an uncountable family of disjoint measurable sets with finite $|\alpha|$-measure. Since $|\alpha|$ is a finite positive measure, this is a contradiction by [40, 215B (i), (ii)]. Therefore all chains are countable.

  We can therefore deduce that each chain has its union as an upper bound, and therefore can apply Zorn's lemma to deduce the existence at least one maximal element in P.

  If $\{S_i\}_{i\in I}$, I countable, is a maximal element in P and $|\alpha|\left(X \setminus \bigcup_{i\in I} S_i\right) > 0$, then there is some $\Sigma \ni S \subseteq X \setminus \bigcup_{i\in I} S_i$ such that $\alpha(S) \neq 0$. By (ii), there is some $F \in \Sigma$ with $\mu(F) < \infty$ such that $\alpha(S \cap F) \neq 0$, and therefore $|\alpha|(S \cap F) > 0$. Now $\mu(S \cap F) < \mu(F) < \infty$, so we can add $S \cap F$ to $\{S_i\}$, contradicting maximality in P.

  We can therefore conclude that any maximal element has $|\alpha|(X \setminus \bigcup_{i\in I} S_i) = 0$, so if $F \cap \bigcup_{i\in I} S_i = \emptyset$, then $|\alpha|(F) = 0$ and so $\alpha(F) = 0$.

- (iii) $\Rightarrow$ (ii):

  Again, countable additivity and absolute continuity are immediately implied. So suppose we have $E \in \Sigma$ and $\alpha(E) \neq 0$. By (iii), we must have that $E \cap \bigcup_{i\in I} S_i \neq \emptyset$.

  We are looking for an $F \in \Sigma$ with $\mu(F) < \infty$ and $\alpha(E \cap F) \neq 0$. If we had $\alpha(E \cap \bigcup_{i=1}^{n} S_i) = 0$ for all n, then we would have $\alpha(E \cap \bigcup_{i=1} S_i) = 0$, so by Lemma 5.4.3 $\alpha(E) \neq 0$. Since this is not so, we must have that there is some $n \in \mathbb{N}$ such that $\alpha(E \cap \bigcup_{i=1}^{n} S_i) \neq 0$. Now

  $$\mu\left(\bigcup_{i=1}^{n} S_i\right) \leq \sum_{i=1}^{n} \mu(S_i) < \infty,$$

so we can define $F = \bigcup_{i=1}^{n} S_i$. □

Let (X, Σ, μ) be a measure space, and recall that $\mathcal{L}^1(X, \Sigma, \mu)$ is the vector space of absolutely integrable $\mathbb{C}$-valued functions, and $L^1(X, \Sigma, \mu)$ the quotient of this by the subspace of functions supported on a set of μ zero. Let $\phi \in L^1(X)$. We define a measure $\phi \cdot \mu$ on X as follows:

$$(\phi \cdot \mu)(S) = \int_X \chi_S \phi d\mu = \int_S \phi d\mu.$$

The relevance of true continuity is the *Radon-Nikodym theorem.*

Theorem 5.4.8 (Radon-Nikodym)**.** *A signed measure ν is truly continuous to μ iff there exists $\phi \in L^1(X, \Sigma, \mu)$ such that $\phi \cdot \mu = \nu$. The element $\phi \in L^1(X, \Sigma, \mu)$ is unique.*

Proof. [40, 232E] for the equivalence and [39, 131H (b)] for the uniqueness. □

We use the notation $\frac{d\nu}{d\mu}$ for the element of $L^1(X, \Sigma, \mu)$ above.

We can define a Boolean algebra $\mathrm{BA}(X, \Sigma, \mu)$ for any measure space as

$$\mathrm{BA}(X, \Sigma, \mu) = \Sigma / \{N \in \Sigma \mid \mu(N) = 0\}.$$

We write [-] for the quotient mapping $\Sigma \to \mathrm{BA}(X, \Sigma, \mu)$, or $[\text{-}]_{\mathrm{BA}}$ when we need to be specific about which quotient mapping.

A measure space is *semifinite* [40, Definition 211F] if for all $S \in \Sigma$ such that $\mu(S) = \infty$, there is a $T \subseteq S$, $T \in \Sigma$, such that $0 < \mu(T) < \infty$. A measure space (X, Σ, μ) is *localizable* [40, Definition 211G] if $\mathrm{BA}(X, \Sigma, \mu)$ is complete and μ is semifinite.

Theorem 5.4.9. *If (X, Σ, μ) is strictly localizable, then it is localizable and so $\mathrm{BA}(X, \Sigma, \mu)$ is a complete Boolean algebra, and $\mu : \mathrm{BA}(X, \Sigma, \mu) \to [0, \infty]$ is completely additive.*

Proof. The fact that every strictly localizable space is localizable is proven in [40, Theorem 211L (d)], and [41, Theorem 322B (e)] shows that localizable, under Fremlin's definition, does imply that $\mathrm{BA}(X, \Sigma, \mu)$ is a complete Boolean algebra. The suprema in $\mathrm{BA}(X, \Sigma, \mu)$ are somewhat complicated to describe, and are given by the definition of H in [40, Theorem 211L (d) (i)]. That μ defines a measure $\mathrm{BA}(X, \Sigma, \mu) \to [0, \infty]$ that is completely additive is shown in [41, Corollary 321D]. □

5.4.2 Measures on Topological and Hyperstonean Spaces

A measure on the Borel sets of a topological space X is called a *Borel measure* on X. A Borel measure on a compact Hausdorff space is *Radon* if it is finite and inner regular with respect to the compact sets.

Theorem 5.4.10 (Riesz Representation Theorem)**.** *Let X be a compact Hausdorff space. The map from Radon Borel measures on X to linear functionals $C(X) \to \mathbb{C}$ defined by*

$$\mu \mapsto \left(a \mapsto \int\limits_X a d\mu \right),$$

where μ is a Radon Borel measure and $a \in C(X)$, is a bijection.

Proof. See [114, Theorem 2.14]. □

We say that a Radon measure on X is normal iff the corresponding linear functional on $C(X)$ is normal. For positive measures on stonean spaces this is equivalent to preservation of bounded directed joins.

The Riesz representation theorem can be combined with Lemma 5.3.5 as follows.

Corollary 5.4.11. *Let A be a complete Boolean algebra. Any normal Radon probability measure μ on the Borel sets of* $\mathrm{Spec}(A)$ *defines a normal state on A by*

$$a \mapsto \mu(\epsilon_A(a)),$$

and the map this defines from normal Radon probability measures to normal states is a bijection. Equivalently, for every stonean space X, the map from normal Radon Borel probability measures on X to normal states on $\mathrm{Clopen}(X)$ *defined by restriction is a bijection.*

Proof. We have the bijection $\iota_{A,\mathbb{C}}$ from normal states on $C(\mathrm{Spec}(A))$ to normal states on A from Lemma 5.3.5, and the Riesz representation theorem (Theorem 5.4.10) is a bijection between normal states on $C(\mathrm{Spec}(A))$ and normal Borel Radon measures on $\mathrm{Spec}(A)$, so their composition is a bijection. If we start with a normal measure μ, the corresponding normal state is given by $a \mapsto \int_X a d\mu$, and applying $\iota_{A,\mathbb{C}}$ to this gives

$$\int_X \chi_{\epsilon_A(a)} d\mu = \mu(\epsilon_A(a)).$$

The corresponding facts for stonean spaces hold by stonean duality (Theorem 5.3.3). □

If A is a complete Boolean algebra, an element $a \in A$ *supports* a normal state ϕ if $\phi(a) = 1$. We can then define

$$\mathrm{supp}(\phi) = \bigwedge \{a \in A \mid \phi(a) = 1\},$$

i.e. to be the meet of all elements supporting ϕ.

Lemma 5.4.12. *Let A be a complete Boolean algebra and ϕ a normal state.*

(i) $\phi(\mathrm{supp}(\phi)) = 1$, i.e. $\mathrm{supp}(\phi)$ *supports ϕ, and therefore* $\mathrm{supp}(\phi)$ *is the smallest element supporting ϕ.*

(ii) For all $a \in A$, $\phi(a) = 0$ implies $a \wedge \mathrm{supp}(\phi) = 0$.

(iii) For all $a \in A$, $\phi(\mathrm{supp}(\phi) \cap a) = \phi(a)$.

Proof.

(i) By Lemma 5.3.4 a normal state preserves directed infima. The set of supports for ϕ is directed because if $\phi(a) = 1$ and $\phi(b) = 1$, then

$$1 = \phi(a) = \phi(a \wedge b) + \phi(a \wedge \neg b)$$

and as $a \wedge \neg b \leq \neg b$, and $\phi(\neg b) = 0$ by finite additivity, $\phi(a \wedge \neg b) = 0$, so $\phi(a \wedge b) = 1$.

Therefore

$$\phi(\mathrm{supp}(\phi)) = \phi\left(\bigwedge\{a \in A \mid \phi(a) = 1\}\right) = \inf_{\{a \in A \mid \phi(a)=1\}} \phi(a) = 1.$$

(ii) Suppose that $\phi(a) = 0$. By finite additivity, $\phi(\neg a) = 1$, so $\mathrm{supp}(\phi) \leq \neg a$. Therefore $a \leq \neg\mathrm{supp}(\phi)$ and so $a \wedge \mathrm{supp}(\phi) = 0$.

(iii) We have $a = (\mathrm{supp}(\phi) \wedge a) \vee (\neg\mathrm{supp}(\phi) \wedge a)$. Since $\phi(\neg\mathrm{supp}(\phi)) = 0$, $\phi(\neg\mathrm{supp}(\phi) \wedge a) = 0$, so by fniinte additivity $\phi(a) = \phi(\mathrm{supp}(\phi) \wedge a)$. □

A family of states $(\phi_i)_{i \in I}$ on a complete Boolean algebra, or a family of normal measures $(\mu_i)_{i \in I}$ on a stonean space X, is said to be *separating* if for any element $a \in A$, $\phi_i(a) = 0$ for all $i \in I$ implies $a = 0$, respectively for any clopen $G \subseteq X$, $\mu_i(G) = 0$ for all $i \in I$ implies $G = \emptyset$.

Proposition 5.4.13. *Let X be a stonean space.*

(i) If μ is a finite normal measure, and $S \subseteq X$ is nowhere dense, then S is a μ-nullset.

(ii) Let $(\phi_i)_{i \in I}$ be a separating family of normal states on Clopen(X) *and μ_i the corresponding family of normal measures under Corollary 5.4.11. Then $S \subseteq X$ is μ_i-null for all $i \in I$ iff S is nowhere dense.*

Proof.

(i) Since S is nowhere dense, $\mathrm{cl}\,(S)$ has empty interior. It suffices to show that $\mu(\mathrm{cl}\,(S)) = 0$. This is defined because $\mathrm{cl}\,(S)$ is a closed, hence Borel, set. Using zero-dimensionality,

$$\mathrm{cl}\,(S) = \bigcap\{G \in \mathrm{Clopen}(X) \mid \mathrm{cl}\,(S) \subseteq G\}.$$

We therefore have

$$\begin{aligned}
\emptyset &= \mathrm{int}\,(\mathrm{cl}\,(S)) \\
&= \mathrm{int}\left(\bigcap\{G \in \mathrm{Clopen}(X) | \mathrm{cl}\,(S) \subseteq G\}\right) \\
&= \bigwedge\{G \in \mathrm{Clopen}(X) \mid \mathrm{cl}\,(S) \subseteq G\} && \text{Lemma 5.2.26 (ii)}
\end{aligned}$$

By Corollary 5.4.11 and Lemma 5.3.4, we have

$$\inf_{\{G \in \mathrm{Clopen}(X) \mid \mathrm{cl}(S) \subseteq G\}} \mu(G) = 0,$$

so for all $\epsilon > 0$, there exists a $G_\epsilon \in \mathrm{Clopen}(X)$ such that $\mathrm{cl}\,(S) \subseteq G_\epsilon$ and $\mu(G_\epsilon) \leq \epsilon$. The set $\bigcap_{i=1}^{\infty} G_{2^{-i}}$ contains $\mathrm{cl}\,(S)$ and has measure zero by [49, Chapter II, §9, Theorem E]. Therefore $\mu(\mathrm{cl}\,(S)) = 0$.

(ii) If S is nowhere dense, we have that it is μ_i-null for all i by part (i). We therefore reduce to showing the other implication, that if S is μ_i-null for all $i \in I$, it is nowhere dense. Suppose for a contradiction that $\mathrm{int}\,(\mathrm{cl}\,(S)) \neq \emptyset$. Since $\mathrm{int}\,(\mathrm{cl}\,(S))$ is the interior of a closed set in a stonean space, it is a clopen set, so its characteristic function is continuous. Since ϕ_i is a separating family, there must be some $i \in I$ such that $\phi_i(\chi_{\mathrm{int}(\mathrm{cl}(S))}) \neq 0$, so $\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S))) \neq 0$.

Now, since S was assumed to be a nullset for all μ_i, we have that there is a Borel set $S' \subseteq X$ such that $S \subseteq S'$ and $\mu_i(S') = 0$. Since μ_i is a Radon measure, we have that for all $\epsilon > 0$, there is a compact $K_\epsilon \subseteq X \setminus S'$ such that $\mu(X \setminus K_\epsilon) \leq \epsilon$. We have $S \subseteq S' \subseteq X \setminus K_\epsilon$, and therefore $\mathrm{int}\,(\mathrm{cl}\,(S)) \subseteq \mathrm{int}\,(\mathrm{cl}\,(X \setminus K_\epsilon))$. As $X \setminus K_\epsilon$ is open, its closure is open as well, so $\mathrm{int}\,(\mathrm{cl}\,(S)) \subseteq \mathrm{cl}\,(X \setminus K_\epsilon)$. Therefore $\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S))) \leq \mu_i(\mathrm{cl}\,(X \setminus K_\epsilon))$, and Lemma 5.2.5 and (i) imply we have $\mu_i(\mathrm{cl}\,(X \setminus K_\epsilon)) = \mu_i(X \setminus K_\epsilon)$. We can take $\epsilon = \frac{1}{2}\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S)))$, and we get

$$\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S))) \leq \mu_i(X \setminus K_\epsilon) \leq \frac{1}{2}\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S))).$$

This contradicts $\mu_i(\mathrm{int}\,(\mathrm{cl}\,(S))) \neq 0$, so our initial assumption that we could have $\mathrm{int}\,(\mathrm{cl}\,(S)) \neq \emptyset$ is refuted. □

5.4.3 Measurable functions and $L^\infty(X, \Sigma, \mu)$

In this chapter, it is important, given measure spaces $(X, \Sigma, \mu), (Y, \Theta, \nu)$, to distinguish measurable *maps* $X \to Y$, from measurable *functions*, which are maps $X \to \mathbb{C}$ or $X \to \mathbb{R}$. We will define a commutative W*-algebra $L^\infty(X, \Sigma, \mu)$ for each measure space, and the elements of $L^\infty(X, \Sigma, \mu)$ will be measurable functions, while measurable maps will produce morphisms *between* W*-algebras.

In both cases, we require that the preimages of measurable sets are measurable. For measurable functions, this is all we require, and in fact no measure is fixed on $\mathbb{R}$ or $\mathbb{C}$. However, for measurable maps, we require that the preimage of a ν-null set is μ-null (we call this property being *null-preserving*). For measurable functions, this ensures we get all elements of $L^\infty(X)$, and for measurable maps, this is necessary for the map $L^\infty(Y) \to L^\infty(X)$ to be well-defined. The latter convention is not standard, but is important for the duality in this chapter.

We first define $\mathcal{L}^\infty(X, \Sigma)$, which does not depend on μ.

$$\mathcal{L}^\infty(X, \Sigma) = \{a : X \to \mathbb{C} \mid a \text{ measurable and bounded}\}.$$

Under pointwise addition and scalar multiplication this is a complex vector space. We can define a norm

$$\|a\| = \sup_{x \in X} |a(x)| \tag{5.4}$$

on $\mathcal{L}^\infty$.

Proposition 5.4.14. *$\mathcal{L}^\infty(X, \Sigma)$ is a commutative C^*-algebra.*

Proof. First, $\ell^\infty(X)$ is a commutative C*-algebra with the same norm. Then $\mathcal{L}^\infty(X, \Sigma)$ is a subring because addition, multiplication and complex conjugation are all continuous and therefore measurable operations on $\mathbb{C}$, and so when applied pointwise they preserve measurability of functions. To show it is a C*-algebra, we only need to show that it is closed. If a_i is a sequence in $\mathcal{L}^\infty(X, \Sigma)$ that converges to $a \in \ell^\infty(X)$, we can deduce that it converges pointwise, and pointwise limits of sequences of measurable functions are measurable, as $\mathbb{C}$ is a metric space [31, Theorem 4.2.2], so $a \in \mathcal{L}^\infty(X)$. □

Sometimes, for instance in [40], one defines $\mathcal{L}^\infty$ including the measure and only requiring that a function be bounded away from a set of measure zero, or allowing functions to be undefined on a set of measure zero. If this is done, $\mathcal{L}^\infty(X, \Sigma, \mu)$ (the measure is required in these cases) will not be a C*-algebra, at least with the norm (5.4), because this is not defined for unbounded functions.

We can define $\mathcal{L}^\infty$ on maps $f : (X, \Sigma) \to (Y, \Theta)$ as , taking $b \in \mathcal{L}^\infty(Y, \Theta)$ as follows

$$\mathcal{L}^\infty(f)(b) = b \circ f$$

Proposition 5.4.15. *The above definition defines a functor $\mathcal{L}^\infty : \mathcal{M}\acute{e}s \to$ $\mathbf{CC^*Alg}^{\mathrm{op}}$.*

Proof. We first show that $b \circ f$ is a bounded measurable function. If $S \subseteq \mathbb{C}$ is a Borel set, we have that $(b \circ f)^{-1}(S) = f^{-1}(b^{-1}(S))$, and $b^{-1}(S) \in \Theta$ as b is measurable, and therefore $f^{-1}(b^{-1}(S)) \in \Sigma$, proving it is a measurable function. Now let α be a bound for b, *i.e.* for all $y \in Y$ we have $|b(y)| \leq b$. Then since $f(x) \in Y$ for all $x \in X$, we have that $|(b \circ f)(x)| = |b(f(x))| \leq \alpha$ for all $x \in X$.

We now show that $\mathcal{L}^\infty(f)$ defines a *-homomorphism. We only give the proof explicitly that it is linear, as the proof that it preserves the -* and multiplication are essentially the same, relying on the pointwiseness of the definitions. Let $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{L}^\infty(Y, \Theta)$ and $x \in X$ in the following

$$\begin{aligned}
\mathcal{L}^\infty(f)(\alpha a + \beta b)(x) &= (\alpha a + \beta b)(f(x)) = \alpha a(f(x)) + \beta b(f(x)) \\
&= \alpha \mathcal{L}^\infty(f)(a)(x) + \beta \mathcal{L}^\infty(f)(b)(x) \\
&= (\alpha \mathcal{L}^\infty(f)(a) + \beta \mathcal{L}^\infty(f)(b))(x).
\end{aligned}$$

We can now see that $\mathcal{L}^\infty$ is a functor for the same reason that $\mathcal{C}(\text{-}, X)$ is a functor in any category $\mathcal{C}$, preservation of identity maps follows from the definition of composition with an identity map, and preservation of composition follows from associativity of composition. □

Recall that a measurable function has a notion of support, where if $a \in \mathcal{L}^\infty(X,\Sigma)$,
$$\operatorname{supp}(a) = a^{-1}(\mathbb{C} \setminus \{0\}) = X \setminus a^{-1}(0),$$
which is necessarily an element of Σ. Inside $\mathcal{L}^\infty(X,\Sigma)$ there is the set $N^\infty(X,\Sigma,\mu)$, defined as
$$N^\infty(X,\Sigma,\mu) = \{a \in \mathcal{L}^\infty(X,\Sigma) \mid \mu(\operatorname{supp}(a)) = 0\}.$$

Lemma 5.4.16. *The set $N^\infty(X,\Sigma,\mu)$ is a closed *-ideal in $\mathcal{L}^\infty(X,\Sigma)$.*

Proof. For $a, b \in \mathcal{L}^\infty(X,\Sigma)$, and in fact in $\ell^\infty(X)$, we have

$$\operatorname{supp}(a+b) \subseteq \operatorname{supp}(a) \cup \operatorname{supp}(b) \tag{5.5}$$
$$\operatorname{supp}(\alpha a) \subseteq \operatorname{supp}(\alpha) \tag{5.6}$$
$$\operatorname{supp}(ab) \subseteq \operatorname{supp}(a) \cap \operatorname{supp}(b) \tag{5.7}$$
$$\operatorname{supp}(a^*) = \operatorname{supp}(a) \tag{5.8}$$
$$\operatorname{supp}\left(\lim_{i\to\infty} a_i\right) \subseteq \bigcup_{i=1}^{\infty} \operatorname{supp}(a_i) \tag{5.9}$$

Now (5.5) and (5.6) show that $N^\infty(X,\Sigma,\mu)$ is a subspace of $\mathcal{L}^\infty(X,\Sigma)$, as the sets of measure zero form an ideal. Equation (5.7) shows that if $a \in \mathcal{L}^\infty(X,\Sigma)$ and $b \in N^\infty(X,\Sigma,\mu)$, then $ab \in N^\infty(X,\Sigma,\mu)$, *i.e.* $N^\infty(X,\Sigma,\mu)$ is an ideal. Then (5.8) shows that this is a *-ideal and (5.9) combined with the fact that sets of measure zero are a σ-ideal shows that $N(X,\Sigma,\mu)$ is closed. □

We can therefore define a C*-algebra $L^\infty(X,\Sigma,\mu) = \mathcal{L}^\infty(X,\Sigma)/N^\infty(X,\Sigma,\mu)$. If $a \in \mathcal{L}^\infty(X,\Sigma)$ we use $[a]$ to represent the equivalence class in $L^\infty(X,\Sigma,\mu)$. The map $[\text{-}] : \mathcal{L}^\infty(X,\Sigma) \to L^\infty(X,\Sigma,\mu)$ is a *-homomorphism. When we need to be specific about which quotient mapping is meant, we use $[\text{-}]_{L^\infty}$ for this one.

If we have $[f] \in L^1(X,\Sigma,\mu)$ and $[g] \in L^\infty(X,\Sigma,\mu)$, then $fg \in \mathcal{L}^1(X,\Sigma,\mu)$ [40, Theorem 243B(e)] and integration can be used to define a bilinear map [40, 234F]
$$([f],[g]) \mapsto \int_X f(x)g(x)d\mu(x), \tag{5.10}$$
and this defines a bijective isometry from $L^1(X)^* \to L^\infty(X)$ if (X,Σ,μ) is localizable [40, Theorems 243G, 243K].

Proposition 5.4.17. *If (X,Σ,μ) is localizable $L^\infty(X,\Sigma,\mu)$ is a commutative W* algebra.*

Proof. By Proposition 5.4.14 and Lemma 5.4.16, $L^\infty(X,\Sigma,\mu)$ is a commutative C*-algebra. Since $L^\infty(X,\Sigma,\mu) \cong L^1(X,\Sigma,\mu)^*$, it is a commutative W*-algebra. □

In fact the notion of localizability was introduced in [119] as a characterization of measure spaces in which $L^\infty(X,\Sigma,\mu)$ was a W*-algebra.

We have seen that $\text{Proj}(C(X)) \cong \text{Clopen}(X)$ and $\text{Proj}(\mathcal{L}^\infty(X,\Sigma)) \cong \Sigma$. Here we give a similar isomorphism for $\text{Proj}(L^\infty(X,\Sigma,\mu))$.

$$s' : \text{Proj}(L^\infty(X,\Sigma,\mu)) \to \text{BA}(X,\Sigma,\mu) \qquad c' : \text{BA}(X,\Sigma,\mu) \to \text{Proj}(L^\infty(X,\Sigma,\mu))$$
$$s'([p]) = [p^{-1}(1)] \qquad c'([S]) = [\chi_S]$$

Proposition 5.4.18. *For any measure space* (X,Σ,μ)*,* s' *and* c'*, as given above, are well defined, mutually inverse, and order preserving, defining an isomorphism* $\text{Proj}(L^\infty(X,\Sigma,\mu)) \cong \text{BA}(X,\Sigma,\mu)$*.*

Proof.

- s' is well-defined and c' is its left inverse:

  Suppose $[p] = [q]$ and both are projections in $L^\infty(X)$. Then we have that $[p^2] = [p]$, and so there is a nullset N such that $p^2|_{X\setminus N} = p|_{X\setminus N}$. Therefore $p = 0$ or $p = 1$ on $X \setminus N$. By extending with zeroes, we can define p' such that $p' = \chi_{p^{-1}(1)}$ and $[p'] = [p] = [q]$. Similarly, we have a q' such that $[q'] = [q]$ and $q' = \chi_{q^{-1}(1)}$. We have that $[q'] = [p']$, so $[q^{-1}(1)] = [p^{-1}(1)]$, and therefore s' is well-defined. We have also shown that $c' \circ s' = \text{id}_{\text{Proj}(L^\infty(X))}$.

- c' is well-defined: Suppose $S, T \in \Sigma$ and $[S] = [T]$, *i.e.* $S \triangle T$ is a nullset. If $x \in X \setminus (S \triangle T)$, then if $x \in S$, we have that $x \in T$, and likewise if $x \notin S$, $x \notin T$. Therefore χ_S and χ_T agree outside the nullset $S \triangle T$, and so $[\chi_S] = [\chi_T]$.

- c' is a right inverse to s': We must show that $s' \circ c' = \text{id}_{\text{BA}(X)}$, which is to say, we must show that if $S \in \Sigma$, we have $s'(c'([S])) = [S]$. Observe:

$$s'(c'([S])) = s'([\chi_S]) = [\chi_S^{-1}(1)] = [S].$$

- c' is monotone: Suppose we have $[S] \le [T]$. Then there exist S' and T' such that $S' \subseteq T'$ and $[S] = [S']$ and $[T] = [T']$. We have that $\chi_{S'} \le \chi_{T'}$ and therefore $[\chi_{S'}] \le [\chi_{T'}]$. Since c' is well-defined, we have $c'([S]) \le c'([T])$.

- s' is monotone: Suppose $[p] \le [q]$. Without loss of generality, we take p and q to only take the values 0 and 1, and hence are characteristic functions of sets we shall call S and T respectively. We have that $[q-p]$ is positive, *i.e.* that $\{x \in X | q(x) - p(x) < 0\}$ is a nullset. Since $q(x) - p(x)$ can only be less than 0 if $q(x) = 0$ and $p(x) = 1$, i.e. if $x \in S \setminus T$, we have that $S \setminus T$ is a nullset. Now $S = (S \setminus T) \cup (S \cap T)$, so we have

$$s'([p]) = [S] = [S \cap T] \le [T] = s'([q]) \qquad \square$$

We define simple functions in $L^\infty(X,\Sigma,\mu)$ to be those of the form

$$\left[\sum_{i\in I} \alpha_i \chi_{S_i}\right]$$

or equivalently

$$\sum_{i\in I} \alpha_i[\chi_{S_i}] = \sum_{i\in I} \alpha_i c'(S_i).$$

for I finite, $\alpha_i \in \mathbb{C}$, and $S_i \in \Sigma$. We call the subspace of such functions $\mathrm{Simp}(X, \Sigma, \mu)$. We see from Proposition 5.4.18 that $\mathrm{Simp}(X, \Sigma, \mu)$ is the span of the projections in $L^\infty(X, \Sigma, \mu)$.

Lemma 5.4.19. $\mathrm{Simp}(X, \Sigma, \mu)$ *is norm dense in* $L^\infty(X, \Sigma, \mu)$

Proof. To simplify certain expressions, we write $A = L^\infty(X, \Sigma, \mu)$. By Proposition 5.2.27 $\mathrm{Spec}(L^\infty(X, \Sigma, \mu))$ is stonean, hence Stone (Lemma 5.2.25), so Proposition 5.2.19 shows that $\mathrm{Simp}(\mathrm{Spec}(A))$ is dense in $C(\mathrm{Spec}(A))$. Since $\mathrm{Simp}(\mathrm{Spec}(A), \mathbb{C})$ is the span of $\mathrm{Proj}(C(\mathrm{Spec}(A)))$ (Corollary 5.2.18), then the Gelfand isomorphism $\epsilon_A : A \to C(\mathrm{Spec}(A))$ maps $\mathrm{Simp}(X, \Sigma, \mu)$ to the (other) simple functions $\mathrm{Simp}(\mathrm{Spec}(A), \mathbb{C})$, so we can conclude that $\mathrm{Simp}(X, \Sigma, \mu)$ is dense in $A = L^\infty(X, \Sigma, \mu)$. Alternatively, one can use [40, 243I]. □

It is convenient to define L^∞ on maps in more generality than we will use when proving it is an equivalence. Let $f : (X, \Sigma, \mu) \to (Y, \Theta, \mu)$ be a measurable map. Given $b \in \mathcal{L}^\infty(Y)$, representing an element $[b] \in L^\infty(Y)$, we define

$$L^\infty(f)([b]) = [b \circ f].$$

Proposition 5.4.20. *For any measurable map* $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ *the above definition of* $L^\infty(f)$ *defines a unital *-homomorphism* $L^\infty(Y, \Theta, \nu) \to L^\infty(X, \Sigma, \mu)$.

Proof. If $[b] \in L^\infty(Y, \Theta, \nu)$, then by Proposition 5.4.15, $b \circ f$ is a bounded measurable function, so $[b \circ f] \in L^\infty(X, \Sigma, \mu)$. We must show that this gives the same equivalence class of functions if we start with equivalent functions, *i.e.* that it is well defined $L^\infty(Y) \to L^\infty(X)$. If $[b] = [b']$, then $b - b' \in N^\infty(Y)$. Since $b \circ f - b' \circ f = (b - b') \circ f$, it suffices to show that $N^\infty(Y) \circ f \subseteq N^\infty(X)$. So let $n \in N^\infty(Y)$, i.e. $\nu(\mathrm{supp}(n)) = 0$. Recall that

$$\mathrm{supp}(n) = \{y \in Y \mid f(y) \neq 0\}.$$

We have that

$$\begin{aligned}\mathrm{supp}(n \circ f) &= \{x \in X \mid n(f(x)) \neq 0\} = \{x \in X \mid f(x) \in \mathrm{supp}(n)\} \\ &= f^{-1}(\mathrm{supp}(n)).\end{aligned}$$

Since f is null-preserving (by the definition of a measurable *map*), we have that $\mu(f^{-1}(\mathrm{supp}(n))) = 0$, so $n \circ f \in N^\infty(X)$ as required.

We give only the proof of linearity explicitly, as the proofs of preservation of $*$, multiplication and unit are similar. They all go through the proof of

Proposition 5.4.15. Let $\alpha, \beta \in \mathbb{C}$ and $[a], [b] \in L^\infty(Y)$. Then

$$\begin{aligned} L^\infty(f)(\alpha[a] + \beta[b]) &= L^\infty(f)([\alpha a + \beta b]) \\ &= [(\alpha a + \beta b) \circ f] \\ &= [\alpha(a \circ f) + \beta(b \circ f)] && \text{Proposition 5.4.15} \\ &= \alpha[a \circ f] + \beta[b \circ f] \\ &= \alpha L^\infty(f)([a]) + \beta L^\infty(f)([b]). \end{aligned}$$

The preservation of identity maps and composition holds by the same argument as for Proposition 5.4.15. □

We can define BA on measurable maps $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ as:

$$\mathrm{BA}(f)([S]) = [f^{-1}(S)]$$

we defer the proof that this is a functor until the next section when L^∞ is proven to be a functor. The following is the L^∞ analogue of Lemma 5.2.15.

Lemma 5.4.21. *Suppose $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ is a measurable map. Let $T \in \Theta$. Then $L^\infty(f)([\chi_T]) = [\chi_{f^{-1}(T)}]$.*

Proof. Expanding the definitions, $L^\infty(f)([\chi_T]) = [\chi_T \circ f] = [\chi_{f^{-1}(T)}]$, using Lemma 5.2.15 (i). □

5.5 The category $\mathcal{Meas}$ and the functor L^∞

In this section, we give the definition of $\mathcal{Meas}$, a category that we will use to define **Meas** and the functor $L^\infty : \mathcal{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$. The objects of $\mathcal{Meas}$ are strictly localizable compact complete measure spaces. We also require that if $\mu(X) = 0$ for the whole space X, then $X = \emptyset$. The definition of the maps requires some care, as we need to be able to prove that $L^\infty(f)$ is a map in $\mathbf{CW^*Alg}$, *i.e.* a normal map of W*-algebras. We say that a function $f : (X, \Sigma, \mu) \to (X, \Theta, \nu)$ is a *normal map* of measure spaces if it is a map of measurable spaces such that if $\alpha : \Sigma \to [0, \infty)$ is truly continuous to μ, then $f_*(\alpha)$ is truly continuous to ν.

We define $\mathcal{Meas}((X, \Sigma, \mu), (Y, \Theta, \nu))$ to be the set of normal maps $(X, \Sigma, \mu) \to (Y, \Theta, \nu)$. We must show that this is a category.

Lemma 5.5.1.

(i) The identity map $\mathrm{id} : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is normal.

(ii) If $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ and $g : (Y, \Theta, \nu) \to (Z, \Phi, \xi)$ are normal, then $g \circ f : (X, \Sigma, \mu) \to (Z, \Phi, \xi)$ is normal.

Proof.

(i) We have that $\mathrm{id}^{-1}(S) = S$ and hence id is measurable. For the same reason, we have that $\mathrm{id}_*(\nu) = \nu$ for all measures ν on (X, Σ), so if ν is truly continuous to μ, so is $\mathrm{id}_*(\nu)$, hence id is normal.

(ii) Suppose $S \in \Phi$. Then $g^{-1}(S) \in \Theta$ and $(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S)) \in \Sigma$, so $g \circ f$ is measurable. Suppose μ' is truly continuous to μ. Then $f_*(\mu')$ is truly continuous to ν, and so $g_*(f_*(\mu'))$ is truly continuous to ξ. We have that $(g \circ f)_*(\mu') = g_*(f_*(\mu'))$, so f is truly continuous. □

The identity and associativity laws then follow because they hold for functions. We did not make $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ being a measurable map part of the definition, only that it be a map of measurable spaces. This is because we have the following result.

Lemma 5.5.2. *If f is a normal map $(X, \Sigma, \mu) \to (Y, \Theta, \nu)$, where (X, Σ, μ) is semifinite (*e.g.* if (X, Σ, μ) is localizable or strictly localizable), it is null-preserving, and therefore a measurable map.*

Proof. We first observe that strictly localizable and localizable imply semifinite by [40, 211L (d),(e)]. Now we deal with the main part of the theorem. Suppose for a contradiction that there exists $S \in \Theta$ with $\nu(S) = 0$ but $\mu(f^{-1}(S)) \neq 0$. Because (X, Σ, μ) is semifinite, there exists $F \in \Sigma$, $F \subseteq f^{-1}(S)$ such that $0 < \mu(F) < \infty$. We have that $\chi_F \in L^1(X)$, and therefore $\chi_F \cdot \mu$ is truly continuous to μ. Since f is normal, we have that $f_*(\chi_F \cdot \mu)$ is truly continuous to ν, so by Theorem 5.4.8 there exists $\psi \in L^1(Y)$ such that $f_*(\chi_F \cdot \mu) = \psi \cdot \nu$. Now we evaluate these measures on S two different ways. For the right hand side we have:

$$(\psi \cdot \nu)(S) = \int_Y \chi_S \psi d\nu$$

Since $\nu(S) = 0$, we have that

$$\int_Y \chi_S \psi d\nu = 0.$$

For the left hand side we have:

$$f_*(\chi_F \cdot \mu)(S) = \int_X \chi_{f^{-1}(S)} \chi_F d\mu = \int_X \chi_F d\mu = \mu(F) > 0,$$

which is a contradiction. □

Given (X, Σ, μ), a set $S \in \Sigma$ is said to be (μ-)σ-finite if there exists a countable family $S_i \in \Sigma$ such that $\mu(S_i) < \infty$ and $\bigcup_{i=1}^{\infty} S_i = S$.

The following proposition is not used in the following, but allows another formulation of the normal maps.

Proposition 5.5.3. *For a measure space (X, Σ, μ), the σ-finite sets form a (possibly trivial) σ-ideal containing the sets of measure zero.*

Proof. We must show that every measurable subset of a σ-finite set is σ-finite and that countable unions of σ-finite sets are σ-finite. The latter follows from the fact that $\mathbb{N} \times \mathbb{N}$ is countable, so we reduce to proving the former. If $S = \bigcup_{i=1}^{\infty} S_i$ for $\mu(S_i) < \infty$, with all these sets being in Σ, and we have $\Sigma \ni T \subseteq S$, we have

$$T = T \cap \bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} T \cap S_i$$

and $\mu(T \cap S_i) \leq \mu(S_i) < \infty$, so T is σ-finite. □

It is important to remark that the μ-σ-finite sets are only necessarily downward closed in Σ, because a σ-finite set may nonetheless contain a non-measurable set. They therefore do not necessarily form a σ-ideal in $\mathcal{P}(X)$.

Lemma 5.5.4. *Let f be measurable. Then f is normal iff f is null-preserving and for all μ-σ-finite $S \in \Sigma$, there exists a ν-σ-finite $T \in \Theta$ such that $\mu(S \setminus f^{-1}(T)) = 0$.*

Proof.

- $\Rightarrow$: Suppose that f is normal. Let $S \in \Sigma$ be μ-σ-finite, so it can be expressed as a disjoint union of μ-finite sets $S = \bigcup_{i \in \mathbb{N}} S_i$ by Lemma 5.2.10. Since each S_i is μ-finite, we have that $\chi_{S_i} \in \mathcal{L}^1(X)$, and so $[\chi_{S_i}] \in L^1(X)$. Therefore $\chi_{S_i} \cdot \mu$ is truly continuous to μ. By the normality of f, we have that $f_*(\chi_{S_i} \cdot \mu)$ is truly continuous to ν. By Definition 5.4.7 (iii), this implies that for each $i \in \mathbb{N}$, $f_*(\chi_{S_i} \cdot \mu)$ has ν-σ-finite support $T_i \in \Theta$. We show that $\mu(S_i \setminus f^{-1}(T_i)) = 0$ for all i, from which it will follow that $\mu\left(S \setminus f^{-1}\left(\bigcup_{i \in \mathbb{N}} T_i\right)\right) = 0$, with $\bigcup_{i \in \mathbb{N}} T_i$ being ν-σ-finite.

  Since T_i is a support for $f_*(\chi_{S_i} \cdot \mu)$, we have

$$\begin{aligned} 0 &= f_*(\chi_{S_i} \cdot \mu)(Y \setminus T_i) = \chi_{S_i} \cdot \mu(f^{-1}(Y \setminus T_i)) = \int_X \chi_{f^{-1}(Y \setminus T_i)} \cdot \chi_{S_i} d\mu \\ &= \int_X \chi_{f^{-1}(Y \setminus T_i) \cap S_i} d\mu = \mu(f^{-1}(Y \setminus T_i) \cap S_i) = \mu((X \setminus f^{-1}(T_i)) \cap S_i) \\ &= \mu(S_i \setminus f^{-1}(T_i)) \end{aligned}$$

- $\Leftarrow$: We now start with a measurable, null-preserving map $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ such that for all μ-σ-finite $S \in \Sigma$, there is a ν-σ-finite $T \in \Theta$ such that $\mu(S \setminus f^{-1}(T)) = 0$.

  Suppose that α is truly continuous to μ. Let $N \in \Theta$ be a ν-null set. Then since f is null-preserving, $\mu(f^{-1}(N)) = 0$. Then $f_*(\alpha)(N) = \alpha(f^{-1}(N)) = 0$ because α is truly continuous to μ, and therefore we have that $f_*(\mu)$ is absolutely continuous to ν. We now use Definition 5.4.7 (iii) for true continuity, so there exists a countable family of μ-finite sets $E_n \in \Sigma$ such that $E = \bigcup_{n=1}^{\infty} E_n$ is a support for α, *i.e.* if $F \in \Sigma$ and

$F \cap E = \emptyset$, $\alpha(F) = 0$. By our initial assumption there exists $E' = \bigcup_{n=1}^{\infty} E'_n$ with $E'_n \in \Theta$ and $\nu(E'_n)$ finite such that $\mu(E \setminus f^{-1}(E')) = 0$. Now

$$\begin{aligned} f_*(\alpha)(Y \setminus E') &= \alpha(f^{-1}(Y \setminus E')) = \alpha(f^{-1}(Y) \setminus f^{-1}(E')) \\ &= \alpha(X \setminus f^{-1}(E')) = \alpha(E \cap (X \setminus f^{-1}(E'))) \\ &= \alpha(E \setminus f^{-1}(E')). \end{aligned}$$

Since $\mu(E \setminus f^{-1}(E')) = 0$ and α is truly continuous to μ, we have $\alpha(E \setminus f^{-1}(E')) = 0$, so we can deduce that $f_*(\alpha)(Y \setminus E') = 0$, and so E' is a ν-σ-finite support for $f_*(\alpha)$, so $f_*(\alpha)$ is truly continuous to ν. □

If, for each measure space, we take the quadruple (X, Σ, N, F), where (X, Σ) are as usual, N is the σ-ideal of sets of measure zero and F the (possibly trivial) σ-ideal of σ-finite sets, each with respect to a fixed measure μ, then we can define normality of a map in terms of these data only, independently of which particular measure that defines the same N and F is used.

We can now see that a measure space is σ-finite in the previously defined sense iff X itself, and hence every element of Σ, is σ-finite.

Corollary 5.5.5. *Suppose Y is σ-finite. Then f is null-preserving iff f is normal.*

Proof. Since Y is σ-finite, then $f^{-1}(Y) = X$ and so if $S \in \Sigma$ is σ-finite, $S \subseteq X = f^{-1}(Y)$. Therefore if f is null-preserving we can conclude by Lemma 5.5.4 that f is normal. If f is normal we have that f is null-preserving by Lemma 5.5.2. □

This of course implies that if X and Y are both probability spaces, the case that holds the most interest, then null-preservation implies normality.

We give a simple example to show that not all measurable maps are normal. Showing that this can hold in strictly localizable spaces is quite involved (it is consistent that it is not so), so we only give the simplest example here, based on [40, 232H (b), (c)].

We require a lemma about true continuity first.

Lemma 5.5.6. *If μ on (X, Σ) is a finite measure, it is truly continuous to itself.*

Proof. We use Definition 5.4.7 (iii). The set X is μ-finite, hence μ-σ-finite, and is a support for μ because $F \cap X = \emptyset \Leftrightarrow F = \emptyset$, and $\mu(\emptyset) = 0$. □

We now give our first, simplest, counterexample to show that not all measurable maps are normal, in which the space (X, Σ, μ) is not localizable.

Counterexample 5.5.7. *Let X be an uncountable set, Σ the σ-algebra of countable and cocountable sets. We have the measure μ that is 1 on cocountable sets and 0 on countable sets, and the counting measure ν. The identity map $f : (X, \Sigma, \mu) \to (X, \Sigma, \nu)$ is null-preserving and measurable, but is not normal.*

Proof. The identity map is measurable as the two σ-algebras are the same. The only nullset for ν is the empty set, which is also null for μ. Since μ is finite, by Lemma 5.5.6 it is truly continuous to itself. Therefore for f to be normal, we need that $f_*(\mu) = \mu$ is truly continuous to ν. However, every σ-finite set for ν is countable, and therefore of μ-measure zero and not a support for μ. □

The above counterexample is amenable to criticism on the grounds that (X, Σ, μ) is not compact and (X, Σ, ν) is not strictly localizable, and we have excluded these spaces from $\mathcal{M}eas$. The following counterexample partially addresses this criticism.

Counterexample 5.5.8. *Let X be an uncountable set admitting a probability measure $\mu : \mathcal{P}(X) \to [0, 1]$ that vanishes on singletons. Take $\Sigma = \mathcal{P}(X)$ and ν to be the counting measure $\mathcal{P}(X) \to [0, \infty]$. Then the identity mapping $f : (X, \Sigma, \mu) \to (X, \Sigma, \nu)$ is null-preserving and measurable, but not normal. The spaces (X, Σ, μ) and (X, Σ, ν) are both strictly localizable.*

Proof. The arguments for measurability and null-preservation are the same as the previous counterexample. As before, since μ is finite it is truly continuous to itself by Lemma 5.5.6. If f were normal, $f_*(\mu) = \mu$ would be truly continuous to ν. However, σ-finite sets for ν are countable, and since μ is σ-additive and vanishes on singletons, it vanishes on countable sets, so cannot have ν-σ-finite support.

The space (X, Σ, μ) is strictly localizable because it is a probability measure and so can be given the trivial decomposition (X). In turn, (X, Σ, ν) is strictly localizable with its decomposition into singletons, using the fact that every set is measurable in Σ. □

The above counterexample relies on the existence of such a set and such a probability measure, equivalent to the existence of a real-valued measurable cardinal [67, Corollary 10.7, Definition 10.8]. We do not know if there is an example where both spaces are compact as well as being strictly localizable under any set-theoretic assumption compatible with the axiom of choice.

To show that, restricted to $\mathcal{M}eas$, L^∞ defines a functor $\mathcal{M}eas \to \mathbf{CW^*Alg}^{\mathrm{op}}$, we will need to show that $L^\infty(f)$ is normal for f a normal map. Recall that for W*-algebras, normality is equivalent to weak-* continuity with respect to the predual (Proposition 5.2.29).

Theorem 5.5.9. *Let $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ be a normal map. The definition*

$$f_{**}(\phi) = \frac{df_*(\phi \cdot \mu)}{d\nu},$$

where $\phi \in L^1(X)$, defines a map $L^1(X) \to L^1(Y)$ such that

$$\int_Y b f_{**}(\phi) d\nu = \int_X L^\infty(f)(b) \phi d\mu$$

for all $b \in L^\infty(Y)$, and so $L^\infty(f)$ is normal. Conversely, if f is only assumed to be measurable and null-preserving, then f is normal if $L^\infty(f)$ is normal.

Proof. First, assume that f is normal. To prove that f_{**} is defined, we reason as follows. Since $\phi \in L^1(X)$, we have that $\phi \cdot \mu$ is truly continuous to μ (Theorem 5.4.8). Since f is normal, we have that $f_*(\phi \cdot \mu)$ is truly continuous to ν, and hence the Radon-Nikodym derivative exists and so f_{**} exists. Similarly, following the definitions we have, for each $T \in \Theta$:

$$\begin{aligned}\int_Y \chi_T f_{**}(\phi) d\nu &= \int_Y \chi_T \frac{df_*(\phi \cdot \mu)}{d\nu} d\nu &&= \int_Y \chi_T df_*(\phi \cdot \mu) = f_*(\phi \cdot \mu)(T) \\ &= (\phi \cdot \mu)(f^{-1}(T)) &&= \int_X \chi_{f^{-1}(T)} \phi d\mu \\ &= \int_X L^\infty(f)([\chi_T]) \phi d\mu, \end{aligned}$$

by Lemma 5.4.21. By linearity of the operations involved, we can deduce that for any simple function $b \in \mathrm{Simp}(Y)$

$$\int_Y b f_{**}(\phi) d\nu = \int_X L^\infty(f)(b) \phi d\mu.$$

We know from (5.10) that the map $L^\infty(X) \to \mathbb{C}$ defined by $\int_X \text{-} \phi d\mu$ for $\phi \in L^1(X)$ is norm-continuous, and $L^\infty(f)$ is norm-continuous because it is a unital *-homomorphism (Proposition 5.4.20 and Lemma 1.2.3), so $\int_X L^\infty(f)(\text{-}) \phi d\mu$ and $\int_Y \text{-} f_{**}(\phi) d\nu$ are norm-continuous. As simple functions are a norm-dense set (Lemma 5.4.19)

$$\int_Y b f_{**}(\phi) d\nu = \int_X L^\infty(f)(b) \phi d\mu$$

for all $b \in L^\infty(Y)$. By Proposition 0.3.3, this implies that $L^\infty(f)$ is $\sigma(L^\infty, L^1)$-continuous, *i.e.* normal.

If, on the other hand, f is assumed to be measurable and null-preserving and $L^\infty(f)$ to be normal, then there exists a $g : L^1(X) \to L^1(Y)$ such that for all $b \in L^\infty(Y)$, $\phi \in L^1(X)$

$$\int_Y b g(\phi) d\nu = \int_X L^\infty(f)(b) \phi d\mu. \tag{5.11}$$

by Proposition 0.3.3.

Suppose that α is truly continuous to μ. We want to show that $f_*(\alpha)$ is truly continuous to ν, and therefore that f is normal. Let $\phi = \frac{d\alpha}{d\mu}$. Then

$$\begin{aligned} f_*(\alpha) &= f_*(\phi \cdot \mu) \\ &= \int_X \chi_{f^{-1}(S)} \phi d\mu \\ &= \int_X L^\infty(f)(\chi_S) \phi d\mu && \text{by Lemma 5.4.21} \\ &= \int_Y \chi_S g(\phi) d\mu && \text{by (5.11)} \\ &= (g(\phi) \cdot \nu)(S). \end{aligned}$$

As $g(\phi) \cdot \nu$ is truly continuous to ν by Theorem 5.4.8, we have shown that f is normal. □

So we can now prove the main result of this section:

Theorem 5.5.10. *L^∞ is a functor $\mathcal{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$.*

Proof. By Theorem 5.5.9, we have that if $f : (X, \Sigma, \mu) \to (Y, \Sigma, \nu)$ in $\mathcal{Meas}$, then $L^\infty(f) : L^\infty(Y) \to L^\infty(X)$ is normal, and by Proposition 5.4.20 it is a *-homomorphism, and therefore a map in $\mathbf{CW^*Alg}$. We therefore only need to show that L^∞ preserves identities and composition. In both cases this is because L^∞ is defined on maps by precomposition, so it preserves identity maps because composing with an identity map is the identity operation, and it is preserves composition of maps because composition of functions is associative. □

We can now show that BA is a functor $\mathcal{Meas} \to \mathbf{CBA}^{\mathrm{op}}$, where $\mathbf{CBA}^{\mathrm{op}}$ is the category of complete Boolean algebras and complete morphisms.

Lemma 5.5.11. BA *is a functor $\mathcal{Meas} \to \mathbf{CBA}^{\mathrm{op}}$ and c' is a natural isomorphism $c' : \mathrm{BA} \Rightarrow \mathrm{Proj} \circ L^\infty$ (and therefore so is s').*

Proof. We first prove that the naturality diagram commutes. The naturality diagram for a normal morphism $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ is

$$\begin{array}{ccc} \mathrm{BA}(Y) & \xrightarrow{c'_Y} & \mathrm{Proj}(L^\infty(Y)) \\ {\scriptstyle \mathrm{BA}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{Proj}(L^\infty(f))} \\ \mathrm{BA}(X) & \xrightarrow[c'_X]{} & \mathrm{Proj}(L^\infty(X)). \end{array}$$

Taking $[T] \in \mathrm{BA}(Y)$, the upper right path reduces to

$$\mathrm{Proj}(L^\infty(f))(c'_Y([T])) = \mathrm{Proj}(L^\infty(f))([\chi_T]) = L^\infty(f)([\chi_T]).$$

The lower left path reduces to

$$c'_Y(\mathrm{BA}(f)([T])) = c'_X([f^{-1}(T)]) = [\chi_{f^{-1}(T)}].$$

These are equal by Lemma 5.4.21.

By Theorem 5.5.9, $L^\infty(f)$ is a normal map of W*-algebras, so $\mathrm{Proj}(L^\infty(f))$ is a complete Boolean algebra homomorphism (see Proposition 5.2.29, (iii) $\Rightarrow$ (i)). We can therefore deduce from $\mathrm{BA}(f) = s'_Y \circ \mathrm{Proj}(L^\infty(f)) \circ c'_X$ and Lemma 5.2.3(iii) that $\mathrm{BA}(f)$ is well-defined and a complete Boolean algebra homomorphism. The commutativity of the naturality diagram also means we can deduce that BA is a functor from the fact that L^∞ is. □

In fact, as Proj of a commutative W*-algebra is always a measure algebra (Corollary 5.3.8), BA is in fact a functor $\mathcal{Meas} \to \mathbf{MeasAlg}^{\mathrm{op}}$.

Corollary 5.5.12. *For a measurable, null-preserving map* $f : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ *between localizable measure spaces,* $\mathrm{BA}(f)$ *is a complete Boolean algebra homomorphism iff* f *is normal.*

Proof. We have that $L^\infty(f)$ is normal iff $\mathrm{BA}(f)$ is a complete Boolean algebra homomorphism by Proposition 5.2.29 and Proposition 5.4.18. By Theorem 5.5.9, $L^\infty(f)$ is normal iff f is normal, so together we have the result. □

The reader might be left wondering why we do not simply allow non-normal maps in $\mathbf{CW^*Alg}$. The reason is that we could not then prove that BA is full.

Counterexample 5.5.13. *For the measure space* $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$*, where the measure* $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ *is the counting measure, there is a Boolean homomorphism* $\phi : \mathrm{BA}(\mathbb{N}) \to 2$ *such that there is no measurable map* $f : 1 \to (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ *such that* $\phi = \mathrm{BA}(f)$*.*

Proof. In this particular case, the only set of measure 0 is $\emptyset$, so the map $[\text{-}] : \mathcal{P}(\mathbb{N}) \to \mathrm{BA}(\mathbb{N})$ is an isomorphism. We take ϕ to be a non-principal ultrafilter $\mathcal{P}(\mathbb{N}) \to 2$, *i.e.* one such that $\phi(\{n\}) = 0$ for all $n \in \mathbb{N}$. If there were a map $p : 1 \to (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, we could define $n = p(*)$ and so

$$\mathrm{BA}(p)(\{n\}) = p^{-1}(\{n\}) = 1,$$

a contradiction. □

5.6 Essential Surjectivity and the Spectrum

The aim of this section is to show that $\mathrm{BA} : \mathcal{M}eas \to \mathbf{MeasAlg}^{\mathrm{op}}$ is essentially surjective. To do this, we need to obtain a measure space from every commutative W*-algebra A. Starting with Stone duality, we can at least obtain a topological space $\mathrm{Spec}(A)$, and the approach is to turn this into a measure space. Recall that the spectrum is defined as

$$\mathrm{Spec}(A) = \{\phi : A \to 2 \mid \phi \text{ a Boolean homomorphism}\}.$$

given the weak-* topology, resulting in a compact Hausdorff space.

Note that these ϕs are not required to be complete homomorphisms. We will see later (Corollary 5.9.2) that the "normal spectrum" is empty whenever there are no points of positive measure, for example for the algebra $\mathrm{BA}([0, 1], \lambda)$, where λ is the Lebesgue measure.

We need to find a suitable measure on $\mathrm{Spec}(A)$. To do this, we use a measure on A. A measure on a complete Boolean algebra A is defined to be a map $\overline{\mu} : A \to [0, \infty]$ such that for any disjoint family $(a_i)_{i \in I}$ of elements of A, we have

$$\overline{\mu}\left(\bigvee_{i \in I} a_i\right) = \sum_{i \in I} \overline{\mu}(a_i),$$

where the sum is interpreted as the supremum of the finite sums (possibly ∞). The theory of measures on measure algebras is analogous to the theory of weights on W*-algebras (see [129, Chapter VII]). A measure $\overline{\mu}$ on a complete Boolean algebra A is

(i) *Semifinite* if for each $a \in A$ such that $\overline{\mu}(a) = \infty$, there exists $b \in A$ such that $0 \neq b \leq a$ and $\overline{\mu}(b) < \infty$.

(ii) *Faithful* if $\overline{\mu}(a) = 0$ implies $a = 0$.

Therefore a complete Boolean algebra A has a faithful semifinite measure $\overline{\mu}$ iff $(A, \overline{\mu})$ is a localizable measure algebra in the sense of Fremlin [41, 321A, 322A (e)].

We will require the following theorem.

Theorem 5.6.1. *Let A be a complete Boolean algebra. The following are equivalent*

(i) A is an object of **MeasAlg**, i.e. *A is separated by normal states.*

(ii) There exists a faithful semifinite measure $\overline{\mu} : A \to [0, \infty]$ expressible as a sum

$$\overline{\mu}(a) = \sum_{i \in I} \phi_i(a)$$

where $(\phi_i)_{i \in I}$ is a family of normal states separating elements of A from 0 and with disjoint supports such that $\bigvee_{i \in I} \operatorname{supp}(\phi_i) = 1$.

(iii) There exists a faithful semifinite measure $\overline{\mu} : A \to [0, \infty]$, i.e. *$(A, \overline{\mu})$ is a localizable measure algebra.*

Proof. We first show that (i) implies (ii). We start with a Zorn's lemma argument. Let P be the poset consisting of families of states F such that $\operatorname{supp}(\phi) \wedge \operatorname{supp}(\psi) = 0$ for all $\phi, \psi \in F$, $\phi \neq \psi$, ordered by $\subseteq$. Let $(F_i)_{i \in I}$ be a chain in P. Then $\bigcup_{i \in I} F_i$ is certainly an upper bound with respect to $\subseteq$ for (F_i), as long as it is an element of P. To show this, let $\phi, \psi \in \bigcup_{i \in I} F_i$, with $\phi \neq \psi$. There exist $i, j \in I$ such that $\phi \in F_i$ and $\psi \in F_j$. As (F_i) is a chain, without loss of generality $F_i \subseteq F_j$, so $\phi \in F_j$ as well, and therefore ϕ, ψ have disjoint support.

By Zorn's lemma, P has maximal elements. If F is such a maximal element, suppose for a contradiction that $S = \bigvee_{\phi \in F} \operatorname{supp}(\phi) \neq 1$. As A is a measure algebra, there exists a normal state ψ on A such that $\psi(\neg S) = 1$ (Lemma 5.3.6 (i)), and therefore $\operatorname{supp}(\psi) \leq \neg S$, and so $F \cup \{\psi\} \in P$, contradicting the maximality of F. Therefore for any maximal $F \in P$, we have $\bigvee_{\phi \in F} \operatorname{supp}(\phi) = 1$.

Define $\overline{\mu} : A \to [0, \infty]$ as

$$\overline{\mu}(a) = \sum_{\phi \in F} \phi(a)$$

If $a \in A$ and $\phi(a) = 0$ for all $\phi \in F$, we have $a \wedge \mathrm{supp}(\phi) = 0$ for all $\phi \in F$ (Lemma 5.4.12 (ii)), so

$$0 = \bigvee_{\phi \in F} a \wedge \mathrm{supp}(\phi) = a \wedge \left(\bigvee_{\phi \in F} \mathrm{supp}(\phi) \right) = a \wedge 1 = a.$$

This shows that F is a separating family. Therefore all we need to do in this part is show that $\overline{\mu}$ is a faithful semifinite measure.

We first show it is a measure. Let $(a_i)_{i \in I}$ be a disjoint family in A. Then

$$\begin{aligned}
\overline{\mu}\left(\bigvee_{i \in I} a_i\right) &= \sum_{\phi \in F} \phi\left(\bigvee_{i \in I} a_i\right) \\
&= \sum_{\phi \in F} \sum_{i \in I} \phi(a_i) && \text{each } \phi \text{ normal} \\
&= \sum_{i \in I} \sum_{\phi \in F} \phi(a_i) && \text{Lemma A.1.6} \\
&= \sum_{i \in I} \overline{\mu}(a_i).
\end{aligned}$$

It is faithful because if $\overline{\mu}(a) = 0$, we have $\sum_{\phi \in F} \phi(a) = 0$. By positivity this implies $\phi(a) = 0$ for all $\phi \in F$, so $a = 0$ because F is separating.

It is semifinite because if $a \in A$ and $\overline{\mu}(a) = \infty$. As F is a separating family, there must be a $\psi \in F$ such that $\psi(a) \neq 0$. Define $b = \mathrm{supp}(\psi) \wedge a$, observing $b \leq a$. For any $\phi \in F$ such that $\phi \neq \psi$, we have

$$\phi(b) = \phi(\mathrm{supp}(\psi) \wedge b) = \phi(\mathrm{supp}(\phi) \wedge \mathrm{supp}(\psi) \wedge b),$$

by Lemma 5.4.12(iii). As $\mathrm{supp}(\phi) \wedge \mathrm{supp}(\psi) = 0$, we have $\phi(b) = 0$ for all $\phi \in F \setminus \{\psi\}$. For ψ itself, we have

$$\psi(b) = \psi(\mathrm{supp}(\psi) \wedge a) = \psi(a)$$

by the same lemma, and as ψ is a state $\psi(a) \leq 1 < \infty$. Therefore

$$\overline{\mu}(b) = \sum_{\phi \in F} \phi(b) = \psi(b) = \psi(a)$$

which is finite and nonzero, as required. So we have shown $\overline{\mu}$ is semifinite and (i) implies (ii).

Now, (ii) implies (iii) *a fortiori*, so we only need to show that (iii) implies (i). We first remark that Fremlin only requires measures on Boolean algebras to be countably additive. But faithful, countably additive measures on complete Boolean algebras are completely additive [41, 322A (e)] so Fremlin's definition of a localizable measure algebra is equivalent to ours. Let $\overline{\mu} : A \to [0, \infty]$ be a faithful semifinite measure, showing $(A, \overline{\mu})$ is a localizable measure algebra. We want to show that for each $a \in A$, such that $a \neq 0$, there is a state ϕ such that $\phi(a) \neq 0$. We first show that, whether or not $\overline{\mu}(a)$ is finite, there is a $b \leq a$ such that $0 < \overline{\mu}(a) < \infty$. If $\overline{\mu}(a)$ is finite, we take $b = a$, and $\overline{\mu}(a) \neq 0$ because $\overline{\mu}$ is faithful. If $\overline{\mu}(a) = \infty$, we use the semifiniteness of $\overline{\mu}$ to obtain such a $b \leq a$. Define, for all $c \in A$:

$$\phi(c) = \frac{\overline{\mu}(b \wedge c)}{\overline{\mu}(b)}.$$

By essentially the same argument as in Lemma 5.3.6 (i), this is a normal state on A with $\phi(b) = 1$. As $b \leq a$, we have

$$\phi(a) = \frac{\overline{\mu}(b \wedge a)}{\overline{\mu}(b)} = \frac{\overline{\mu}(b)}{\overline{\mu}(b)} = 1.$$

□

We will use a faithful semifinite measure to define a compact complete strictly localizable measure space structure on $X = \mathrm{Spec}(A)$ so that $\mathrm{BA}(X, \Sigma, \mu) \cong A$ in **MeasAlg**. By Corollary 5.4.11 we have that for each of the ϕ_i we can define a finite Radon Borel measure on X, which we shall call μ_i, such that $\mu_i(\epsilon_A(a)) = \phi_i(a)$ for all $a \in A$.

We define a measure on $\mathcal{B}o(X)$ as

$$\mu(S) = \sum_{i \in I} \mu_i(S) = \sup_{J \Subset I} \sum_{i \in J} \mu_i(S)$$

Until the end of this section, we will be considering a fixed (but arbitrary) measure algebra A, its spectrum $X = \mathrm{Spec}(A)$, a fixed faithful semifinite measure $\overline{\mu}$ with decomposition $(\phi_i)_{i \in I}$ with supports $(p_i)_{i \in I}$, and the measure μ on $(X, \mathcal{B}o(X))$ defined as above from the corresponding μ_i. We show that μ is a measure:

Proposition 5.6.2. *The function μ is a measure, i.e. it is countably additive, and for any $a \in A$, we have*

$$\mu(\epsilon_A(a)) = \overline{\mu}(a),$$

where $\epsilon_A : A \to C(\mathrm{Spec}(A))$ is the counit of Stone duality.

Proof. Let $\{S_j\}$ be a sequence of disjoint Borel subsets of X.

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} S_j\right) &= \sum_{i \in I} \mu_i \left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{i \in I} \sum_{j=1}^{\infty} \mu_i(S_j) \\
&= \sum_{j=1}^{\infty} \sum_{i \in I} \mu_i(S_j) && \text{Lemma A.1.6} \\
&= \sum_{j=1}^{\infty} \mu(S_j).
\end{aligned}$$

So μ is a measure. Now, consider some $a \in A$.

$$\begin{aligned} \mu(\epsilon_A(a)) &= \sum_{i \in I} \mu_i(\epsilon_A(a)) \\ &= \sum_{i \in I} \phi_i(a) && \text{Corollary 5.4.11} \\ &= \overline{\mu}(a). \end{aligned}$$

□

We take our measure space to be $(X = \mathrm{Spec}(A), \hat{\Sigma}, \hat{\mu})$ where $\hat{\Sigma}$ is the μ-measurable sets obtained by completing with respect to the original μ on the Borel sets Σ of X. We still need to show that it is strictly localizable, compact, and that $\mathrm{BA}(X, \hat{\Sigma}, \hat{\mu}) \cong A$.

Proposition 5.6.3. *The following four statements about a subset $S \subseteq X$ are equivalent:*

(i) S is a $\hat{\mu}$-nullset.

(ii) S is a $\hat{\mu}_i$-nullset for all $i \in I$.

(iii) S is nowhere dense.

(iv) S is meagre.

Proof.

- (iii) ⇔ (ii): This is simply Proposition 5.4.13.
- (ii) implies (i): Suppose S is a $\hat{\mu}_i$-nullset for all $i \in I$. Then

$$\hat{\mu}(S) = \sum_{i \in I} \hat{\mu}_i(S) = \sum_{i \in I} 0 = 0$$

  and hence S is a $\hat{\mu}$-nullset.
- (i) implies (ii): Let S be a $\hat{\mu}$-nullset, which is to say $S \subseteq B$ a Borel set of X, such that $\mu(B) = 0$. Then

$$0 = \mu(B) = \sum_{i \in I} \mu_i(B),$$

  and since this is a sum of positive terms, each $\mu_i(B) = 0$, and so S is a μ_i-nullset for all $i \in I$.
- (iii) ⇔ (iv): By definition every nowhere dense set is meagre. If a set is meagre, then it is a countable union of nowhere dense sets, and therefore a countable union of $\hat{\mu}$-nullsets, and therefore a $\hat{\mu}$-nullset, and therefore nowhere dense. □

Since nullsets for any countably additive measure form a σ-ideal, this implies that every meagre set is nowhere dense.

Corollary 5.6.4. *The set $\hat{\Sigma}$ is exactly those sets with the Claire property, or the sets symmetrically differing from a clopen set by a $\hat{\mu}$-nullset.*

Proof. If $S \subseteq X$ has the Claire property, then there is a clopen G such that $S \triangle G$ is meagre. By Proposition 5.6.3, being meagre is equivalent to being a $\hat{\mu}$-nullset. Since a clopen set is Borel, we have that S is μ-measurable.

If S instead is μ-measurable, then there is a Borel set B such that $S \triangle B$ is μ-null, so by Proposition 5.6.3 $S \triangle B$ is meagre. We have that B has the Baire property (Proposition 5.2.9), so since X is stonean, it has the Claire property (Corollary 5.2.33), so there is a G such that $B \triangle G$ is meagre. Since $\sim$ is an equivalence relation (Proposition 5.2.1), we have $S \triangle G$ is meagre, so S has the Claire property. □

We have therefore shown that $(X, \hat{\Sigma}, \hat{\mu})$ agrees with Fremlin's definition of the Stone space of a localizable measure algebra [41, Definition 321K] where the σ-algebra is defined to be the Claire sets.

Given the support projections p_i for the functionals ϕ_i that make up the measure $\overline{\mu} : A \to [0, \infty]$, we have a family of clopen supports $(S_i)_{i \in I}$ for $(\mu_i)_{i \in I}$, where $S_i = \epsilon_A(p_i)$ (Lemma 5.4.12 and Corollary 5.4.11). Since $\bigvee_{i \in I} p_i = 1$, we can deduce from Lemma 5.2.3 that $\bigvee_{i \in I} S_i = X$. This implies $\mathrm{cl}\left(\bigcup_{i \in I} S_i\right) = X$ (Lemma 5.2.26(i)), so $S' = X \setminus \bigcup_{i \in I} S_i$ is a closed set of empty interior, and therefore nowhere dense. By Proposition 5.6.3, S' is a nullset (for μ and all the μ_i). We pick an element $x \in I$ and define $T_x = S' \cup S_x$ and $T_i = S_i$ otherwise. Then $(T_i)_{i \in I}$ is a family of closed (hence Borel) sets that form a partition of X, and $\mu(T_i) = \mu(S_i)$ for all $i \in I$. We can now show that $(T_i)_{i \in I}$ is a decomposition of $(X, \hat{\Sigma}, \hat{\mu})$.

Proposition 5.6.5. *The measure space $(X, \hat{\Sigma}, \hat{\mu})$, with $\hat{\Sigma}$ the measurable sets, is strictly localizable with decomposition $\{T_i\}_{i \in I}$.*

Proof. We show that each T_i has finite measure as follows. We have that $\mu(T_i) = \mu(S_i)$ for all i by construction, and $\mu(S_i) = \mu(\epsilon_A(p_i)) = \overline{\mu}(p_i)$ by Proposition 5.6.2. Then

$$\overline{\mu}(p_i) = \overline{\mu}(\mathrm{supp}(\phi_i)) = \sum_{j \in I} \phi_j(\mathrm{supp}(\phi_i)) = \phi_i(\mathrm{supp}(\phi_i)) = 1$$

by the disjointness of supports and 5.4.12, proving the finiteness.

We need to show that given a set $S \subseteq X$, if $S \cap T_i \in \hat{\Sigma}$ for all $i \in I$, then $S \in \hat{\Sigma}$. Let S be such a set. We define $U_i = S \cap S_i$. Being clopen, each S_i is a hyperstonean space in its own right (Lemma 5.2.36). The singleton $\{\mu_i\}$ is a separating family of measures on S_i, because if $\mu_i(G) = 0$ for a clopen subset of S_i, then $S_i \cap G = \emptyset$ (Lemma 5.4.12 (ii)), so $G = \emptyset$. By Proposition 5.4.13 each U_i symmetrically differs from a clopen set V_i by a nowhere dense set $N_i = U_i \triangle V_i$.

By definition, $N_i \subseteq S_i$, so by Lemma 5.2.8 we have that

$$N' = \bigcup_{i \in I \setminus \{x\}} N_i$$

is nowhere dense, where x is the distinguished element used to define $(T_i)_{i \in I}$. We define U', V' analogously, taking unions over everything in I but x. By Lemma 5.2.4, we have that

$$U' \triangle V' = N'$$

with the disjointness criterion being implied by the disjointness of the S_i. Since V' is a union of open sets, it is open, hence Borel, and hence U' is μ-measurable. Since $S \cap T_x$ is measurable and measurable sets are a σ-algebra, we have that $(S \cap T_x) \cup U' = (S \cap T_x) \cup \bigcup_{i \in I \setminus \{x\}} S \cap T_i = S$ is measurable, as required.

Now we must show that for all measurable sets $S \subseteq X$

$$\hat{\mu}(S) = \sum_{i \in I} \hat{\mu}(S \cap T_i)$$

We have that there is a Borel set B and a μ-nullset N such that $S \triangle B = N$ and hence

$$\begin{aligned} \hat{\mu}(S) = \hat{\mu}(B) = \mu(B) &= \sum_{i \in I} \mu_i(B) \\ &= \sum_{i \in I} \mu_i(B \cap T_i) \qquad \text{by Lemma 5.4.3 and 5.4.12 and } S_i \subseteq T_i \end{aligned}$$

Now we consider $\mu(B \cap T_i)$. By definition

$$\mu(B \cap T_i) = \sum_{j \in I} \mu_j(B \cap T_i) = \sum_{j \in I} \mu_j(B \cap T_i \cap T_j) = \mu_i(B \cap T_i).$$

Putting these three facts together, we have

$$\hat{\mu}(S) = \sum_{i \in I} \mu(B \cap T_i).$$

We know that $S \triangle B = N$, a μ-nullset, *i.e.*

$$N = (S \setminus B) \cup (B \setminus S).$$

We have that

$$\begin{aligned} N \cap T_i = ((S \setminus B) \cup (B \setminus S)) \cap T_i &= ((S \setminus B) \cap T_i) \cup ((B \setminus S) \cap T_i) \\ &= ((S \cap T_i) \setminus (B \cap T_i)) \cup ((B \cap T_i) \setminus (S \cap T_i)) \\ &= (S \cap T_i) \triangle (B \cap T_i). \end{aligned}$$

Since $N \cap T_i$ is a subset of a μ-nullset, it is a μ-nullset, and since it is a μ-nullset. Therefore $\hat{\mu}(S \cap T_i) = \hat{\mu}(B \cap T_i)$, and putting it all together we have

$$\hat{\mu}(S) = \sum_{i \in I} \hat{\mu}(S \cap T_i).$$

for all μ-measurable sets S. We have verified all three conditions for a measure space to be strictly localizable. □

We show that μ is inner regular with respect to the family compact sets of $\mathcal{K} \subseteq \mathcal{P}(X)$. It will follow from [41, Proposition 342G (b)] that $\hat{\mu}$ is inner regular with respect to $\mathcal{K}$, and $\mathcal{K}$ is a compact class by the finite intersection characterization of compactness.

Lemma 5.6.6. *Let B be a Borel subset of X such that $\mu(B) > 0$, and $\alpha < \mu(B)$. Define $B_i = B \cap T_i$. There exists a $J \Subset I$ such that*

$$\mu\left(\bigcup_{i \in J} B_i\right) > \alpha$$

Proof. Since $\mu(S) = \hat{\mu}(S)$ for all Borel sets S by definition, and because $\hat{\mu}$ is strictly localizable with decomposition $\{T_i\}_{i \in I}$ (Proposition 5.6.5), we have

$$\mu(B) = \hat{\mu}(B) = \sum_{i \in I} \hat{\mu}(B \cap T_i) = \sum_{i \in I} \mu(B \cap T_i) = \sup_{J \Subset I} \sum_{i \in J} \mu(B \cap T_i).$$

Since $\alpha < \mu(B)$, it cannot be an upper bound, so there exists some $J \Subset I$ such that

$$\alpha < \sum_{i \in J} \mu(B \cap T_i) = \mu\left(\bigcup_{i \in J} B \cap T_i\right),$$

with the last equality being because the T_i are disjoint. □

Theorem 5.6.7. *The measure μ on the Borel sets of X is inner regular with respect to the compact subsets of X.*

Proof. We need to show that for all Borel subsets B of X,

$$\mu(B) = \sup_{K \subseteq B} \mu(K) \tag{5.12}$$

with K always being compact. We first dispose of the case $\mu(B) = 0$. In that case, since $K \subseteq B$ and the measure is positive, we have $\mu(K) = 0$, so the supremum on the right is also 0, as required. Therefore we will assume from now on that $\mu(B) > 0$, possibly infinite.

The right hand side of (5.12) cannot exceed the left hand side because $\mu(K) \leq \mu(B)$ because it is a subset. Therefore it suffices to show that it is the *least* such upper bound, *i.e.* that if $\alpha < \mu(B)$, then it is not an upper bound because there exists a compact set $K \subseteq B$ such that $\mu(K) \geq \alpha$.

So let $0 < \alpha < \mu(B)$. If $\mu(S)$ is finite, take

$$\beta = \frac{\alpha + \mu(B)}{2}$$

and if $\mu(B) = \infty$, take
$$\beta = \alpha + 1.$$
In either case, we have
$$\alpha \leq \beta < \mu(B)$$
By Lemma 5.6.6, we can find a finite set $J \Subset I$ such that
$$\sum_{i \in J} \mu_i(B_i) = \mu\left(\bigcup_{i \in J} B_i\right) > \beta.$$
Since $\beta \geq \alpha > 0$ we have that J is not empty.

Now define
$$\epsilon = \frac{\beta - \alpha}{2|J|},$$
noting that we (at first) allow $\epsilon = 0$. We deal with each case separately.

- $\epsilon \neq 0$: Since each of the μ_is is inner regular, we have for each $i \in J$, a $K_i \subseteq B_i$ such that $\mu_i(B_i) - \mu(K_i) < \epsilon$. Being contained in disjoint sets, the K_i are disjoint. We define
$$K = \bigcup_{i \in J} K_i,$$
which is compact as it is a finite union of compact sets.

  Adding together the inequalities, we have that:
$$\sum_{i \in J} \mu_i(B_i) - \sum_{i \in J} \mu_i(K_i) < |J|\epsilon = \frac{\beta - \alpha}{2}.$$
  So we have
$$\begin{aligned}\mu(K) \geq \sum_{i \in J} \mu_i(K) &> \sum_{i \in J} \mu_i(B_i) - \frac{\beta - \alpha}{2} \\ &\geq \beta - \frac{\beta - \alpha}{2} && \text{by definition of } J \\ &= \frac{\alpha + \beta}{2} \geq \alpha.\end{aligned}$$
  Taken together, this shows that α is not an upper bound for $\mu(K)$ as K ranges over compact subsets of B, as required.

- $\epsilon = 0$: If $\epsilon = 0$, we have that $\beta = \alpha$. This cannot happen if $\mu(B) = \infty$, as the difference between α and β is 1 by definition in this case. Therefore $\mu(B) < \infty$. By substituting α into the definition of β, we have
$$\alpha = \frac{\alpha + \mu(B)}{2} \Leftrightarrow 2\alpha = \alpha + \mu(B) \Leftrightarrow \alpha = \mu(B)$$
  which contradicts our choice that $\alpha < \mu(B)$. Therefore $\epsilon = 0$ cannot occur in this case, either. □

Corollary 5.6.8. *The space $(X, \hat{\Sigma}, \hat{\mu})$ is a compact measure space, taking the compact subsets of X as the compact class.*

Proof. By [41, Proposition 342G (b)], if a measure space (X, Σ, μ) is compact with respect to a compact class $\mathcal{K}$, then the completion $(X, \hat{\Sigma}, \hat{\mu})$ is compact with respect to the same class $\mathcal{K}$. Therefore we only need apply this result to the previous theorem. □

All together, we have that $(\mathrm{Spec}(A), \hat{\Sigma}, \hat{\mu})$ is a compact complete strictly localizable measure space, and hence an object of $\mathcal{M}eas$. We now show that $\mathrm{BA}(\mathrm{Spec} A) \cong A$. We have the obvious map

$$\begin{aligned} \epsilon'_A &: A \to \mathrm{BA}(\mathrm{Spec}(A)) \\ \epsilon'_A(a) &= [\epsilon_A(a)], \end{aligned}$$

Theorem 5.6.9. *The functor* $\mathrm{BA} : \mathcal{M}eas \to \mathbf{MeasAlg}^{\mathrm{op}}$ *is essentially surjective.*

Proof. Given $A \in \mathbf{MeasAlg}$, we put a faithful measure on it using Theorem 5.6.1. We use this measure and a decomposition of it as $\overline{\mu} = \sum_{i \in I} \phi_i$ to form a measure μ on $(\mathrm{Spec}(A), \mathcal{B}o(\mathrm{Spec}(A)))$ and a decomposition of the measure space $(\mathrm{Spec}(A), \widehat{\mathcal{B}o(\mathrm{Spec}(A))}, \hat{\mu})$ as $(T_i)_{i \in I}$ (Proposition 5.6.5), and this measure space is compact by Corollary 5.6.8. It is complete by construction as a completion. If $\mu(\mathrm{Spec}(A)) = 0$, then $\overline{\mu}(1) = 0$, so by faithfulness of $\overline{\mu}$ we have $0 = 1$ and $\mathrm{Spec}(A) = \emptyset$. This allows us to conclude that $(X, \widehat{\mathcal{B}o(\mathrm{Spec}(A))}, \hat{\mu}) \in \mathcal{M}eas$, and $\widehat{\mathcal{B}o(\mathrm{Spec}(A))} = \mathcal{C}\ell(\mathrm{Spec}(A))$ (Corollary 5.6.4). The map $\epsilon'_A : A \to \mathrm{BA}(\mathrm{Spec}(A))$ is an order isomorphism by Stone duality and Lemma 5.2.34, and therefore a complete Boolean algebra isomorphism (Lemma 5.2.3), and so an isomorphism in **MeasAlg**. □

5.7 Fullness

In this section we outline Fremlin's proof that BA is full. This is where we see where strict localizability and compactness are used.

5.7.1 Liftings

A *lifting* for a measure space (X, Σ, μ) is a Boolean algebra homomorphism that is a section of the quotient map $[\text{-}]_{\mathrm{BA}} : \Sigma \to \mathrm{BA}(X, \Sigma, \mu)$. There is also a notion of a lifting for measurable functions [130, IV.2, Theorem 3], which is useful in the case of W*-algebras, but there is no benefit in using it in the case of measure algebras. For the first part of the fullness proof we require an important fact about strictly localizable spaces known as the lifting theorem.

Theorem 5.7.1 (Lifting Theorem)**.** *If (X, Σ, μ) is complete and strictly localizable and $\mu(X) > 0$, then it has a lifting.*

Proof. See [41, Theorem 341K]. □

Using a lifting σ, we can define a map $\zeta_X : X \to \mathrm{Spec}(\mathrm{BA}(X))$ as follows, with $x \in X$ and $a \in \mathrm{BA}(X)$:

$$\zeta(x)(a) = x \in \sigma(a),$$

where $x \in \sigma(a)$ is considered to take values in 2. In other words, we use the lifting to pick a set representing a, then check whether x is in it or not.

Theorem 5.7.2. *The map* $\mathrm{BA}(\zeta_X)$ *is an inverse to* $\epsilon'_{\mathrm{BA}(X)}$. *It follows that* $\mathrm{BA}(\zeta_X)$ *is normal and so* ζ_X *is normal.*

Proof. We use [41, Proposition 341P] (since [40, Proposition 213H] shows that (X, Σ, μ) is locally determined). This states that there is a bijection between liftings $\sigma : \mathrm{BA}(X) \to \Sigma$ and functions $f : X \to \mathrm{Spec}(\mathrm{BA}(X))$, such that $[f^{-1}(\epsilon_{\mathrm{BA}(X)}(a))] = a$ for all $a \in \mathrm{BA}(X)$, defined in one direction by $\sigma(a) = f^{-1}(\epsilon_{\mathrm{BA}(X)}(a))$. We have

$$\begin{aligned}\zeta_X^{-1}(\epsilon_{\mathrm{BA}(X)}(a)) &= \{x \in X \mid \zeta_X(x) \in \epsilon_{\mathrm{BA}(X)}(a)\} = \{x \in X \mid \zeta_X(x)(a) = 1\} \\ &= \{x \in X \mid x \in \sigma(a)\} = \sigma(a),\end{aligned}$$

so ζ_X is such a function, and it exists because a lifting exists using the strict localizability of X and Theorem 5.7.1. Then part (b) shows that ζ_X is inverse measure-preserving, and therefore the preimage of any nullset in $\mathrm{Spec}(\mathrm{BA}(X))$ is a nullset in X. Therefore, if T is a measurable subset of $\mathrm{Spec}(\mathrm{BA}(X))$, then there exists a clopen G such that $T \triangle G$ is a nullset by Proposition 5.6.3 and Corollary 5.6.4. We have that $\zeta_X^{-1}(T \triangle G) = \zeta_X^{-1}(T) \triangle \zeta_X^{-1}(G)$, and that this is a nullset, so the completeness of X and the measurability of $\zeta_X^{-1}(G)$ shows T is measurable, and therefore that ζ_X is measurable. As it is nullset preserving, it is also a measurable map.

Then from $[\zeta_X^{-1}(\epsilon_{\mathrm{BA}(X)}(a))] = a$, we deduce

$$a = [\zeta_X^{-1}(\epsilon_{\mathrm{BA}(X)}(a))] = \mathrm{BA}(\zeta_X)([\epsilon_{\mathrm{BA}(X)}(a)]) = \mathrm{BA}(\zeta_X)(\epsilon'_{\mathrm{BA}(X)}(a)),$$

i.e. $\mathrm{BA}(\zeta_X) \circ \epsilon'_{\mathrm{BA}(X)} = \mathrm{id}_{\mathrm{BA}(X)}$. We can conclude, therefore, that $\mathrm{BA}(\zeta_X)$ is a left inverse to $\epsilon'_{\mathrm{BA}(X)}$. Since $\epsilon'_{\mathrm{BA}(X)}$ is an isomorphism, it is a bijection, and so every left inverse is an inverse. Therefore $\mathrm{BA}(\zeta_X)$ is a complete homomorphism by Lemma 5.2.3, and therefore ζ_X is normal by Corollary 5.5.12. □

5.7.2 Compactness

In this section we show where the compactness of a measure space is used to prove fullness. If $\phi \in \mathrm{Spec}(\mathrm{BA}(X))$ then we define a family of subsets of $\mathcal{K}$ as follows

$$\mathcal{K}_\phi = \{K \in \mathcal{K} | \phi([\chi_K]) = 1\}.$$

This family may well be empty for some ϕ. We take the set-theoretic meet of all the elements of $\mathcal{K}_\phi$, *i.e.* X itself if $\mathcal{K}_\phi$ is empty, and the intersection of all elements of $\mathcal{K}_\phi$ otherwise. We define a map $\xi_X : \mathrm{Spec}(\mathrm{BA}(X)) \to X$ by mapping ϕ to a point chosen arbitrarily from the meet of $\mathcal{K}_\phi$.

Theorem 5.7.3. *The map ξ_X is a normal map, with $\mathrm{BA}(\xi_X) = \epsilon'_{\mathrm{BA}(X)}$, and if such a map exists for a strictly localizable measure space (X, Σ, μ) it is compact.*

If (X, Σ, μ) and (Y, Θ, ν) are measure spaces in $\mathcal{M}eas$, and $g : \mathrm{BA}(Y) \to \mathrm{BA}(X)$ is a complete Boolean homomorphism, then $f = \xi_Y \circ \mathrm{Spec}(g) \circ \zeta_X$ is a normal map $X \to Y$ such that $\mathrm{BA}(f) = g$. Therefore the functor $\mathrm{BA} : \mathcal{M}eas \to$ $\mathbf{MeasAlg}^{\mathrm{op}}$ is full.

Proof. Fremlin's [41, Theorem 343B] (i) $\Rightarrow$ (iv) shows that for each $S \in \Sigma$, $\xi_X^{-1}(S) \triangle \epsilon_{\mathrm{BA}(X)}([S])$ is a nullset in $\mathrm{Spec}(\mathrm{BA}(X))$. Since $\epsilon_{\mathrm{BA}(X)}([S])$ is a clopen set, $\xi_X^{-1}(S)$ is a measurable set, so ξ_X is measurable. If $N \in \Sigma$ such that $\mu(N) = 0$, we have that $[N] = 0$, so $\epsilon_{\mathrm{BA}(X)}([N]) = \emptyset$. Therefore $\xi_X^{-1}(N) \triangle \epsilon_{\mathrm{BA}(X)}([N]) = \xi_X^{-1}(N)$, so ξ_X is nullset preserving and therefore a measurable map.

The fact that $\xi_X^{-1}(S) \triangle \epsilon_{\mathrm{BA}(X)}([S])$ is negligible implies that $[\xi_X^{-1}(S)] = [\epsilon_{\mathrm{BA}(X)}([S])]$. Expanding definitions we have

$$\mathrm{BA}(\xi_X)([S]) = [\xi_X^{-1}(S)] = [\epsilon_{\mathrm{BA}(X)}([S])] = \epsilon'_{\mathrm{BA}(X)}([S]),$$

and therefore $\mathrm{BA}(\xi_X) = \epsilon'_{\mathrm{BA}(X)}$. Therefore ξ_X is normal (Corollary 5.5.12).

If (X, Σ, μ) is a strictly localizable measure space such that there exists a measurable map $f : \mathrm{Spec}(\mathrm{BA}(X)) \to X$ such that $\mathrm{BA}(f) = \epsilon'_{\mathrm{BA}(X)}$, then by (iv) $\Rightarrow$ (i) of [41, Theorem 343B] (X, Σ, μ) is locally compact, and so is compact by [41, Proposition 342H (b)].

Now, let (X, Σ, μ) and (Y, Θ, ν) be objects of $\mathcal{M}eas$, and $g : \mathrm{BA}(Y) \to \mathrm{BA}(X)$ be a morphism of measure algebras, *i.e.* a complete Boolean homomorphism. By Proposition 5.3.2 and Stone duality, the map $\mathrm{Spec}(g)$ is continuous and preimages of meagre sets are meagre. By Proposition 5.6.3 this implies that the preimage of a nullset is a nullset, and continuity implies Borel measurability. If T is a measurable subset of $\mathrm{Spec}(\mathrm{BA}(Y))$, then there is a Borel set T' such that $T \triangle T'$ is a nullset. Since taking the preimage is a Boolean homomorphism, the preimage of T' symmetrically differs from a Borel set by a nullset as well, and is therefore measurable, proving that $\mathrm{Spec}(g)$ is measurable and null-preserving map.

We define $f = \xi_Y \circ \mathrm{Spec}(g) \circ \zeta_X$, which is a measurable map. Then [41, Theorem 343B] (iv) $\Rightarrow$ (v) proves that $\mathrm{BA}(f) = g$, essentially by using $\mathrm{BA}(\zeta_X) = (\epsilon'_X)^{-1}$ (Theorem 5.7.2) and $\mathrm{BA}(\xi_Y) = \epsilon'_Y$, and the naturality diagram from ϵ'. By Corollary 5.5.12 f is normal, and therefore a map in $\mathcal{M}eas$, and so BA is full. □

5.8 Faithfulness and Meas

Recall the following proposition from elementary category theory.

Proposition 5.8.1. *A fully faithful functor reflects isomorphisms.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, and $f : X \to Y$ an arrow in $\mathcal{C}$ such that $F(f)$ is an isomorphism with inverse $F(f)^{-1}$. Since F is full, there exists $g : Y \to X$ such that $F(g) = F(f)^{-1}$. We have then that

$$F(g) \circ F(f) = \mathrm{id}_{F(X)} F(g \circ f) = F(\mathrm{id}_X)$$
$$g \circ f = \mathrm{id}_X \qquad \text{since } F \text{ is faithful.}$$

Similarly, $f \circ g = \mathrm{id}_Y$ and so g is an inverse for f. □

Proposition 5.8.2. $\mathrm{BA} : \mathcal{M}eas \to \mathbf{MeasAlg}^{\mathrm{op}}$ *is not faithful.*

Proof. In [41, Example 343I], Fremlin gives an example of a map f from a measure space (X, Σ, μ) to itself such that $f(x) \neq x$ for all x in the measure space, but $\mu(f^{-1}(S) \triangle S) = 0$ for all $S \in \Sigma$. In particular this means that $[f^{-1}(S)] = [S]$, *i.e.* that $\mathrm{BA}(f) = \mathrm{id}_{\mathrm{BA}(X)} = \mathrm{BA}(\mathrm{id}_X)$, but $\mathrm{id}_X \neq f$, and in fact, they are not even equal almost everywhere for μ. The measure space X is a probability measure space, hence strictly localizable, and is a Radon measure, so is compact, and can be taken to be complete. We can also see that the property we observed above implies that f is null-preserving, and so the σ-finiteness of X implies that f is a normal map *i.e.* a map in $\mathcal{M}eas$ (Corollary 5.5.5). Therefore BA is not faithful, by Proposition 5.8.1. □

It is also possible to given an example based on the map $2^\omega \to [0, 1]$ mapping each binary decimal to the real number it defines. This does not give an example with the stronger property that two maps that do not agree almost everywhere define the same morphism of W*-algebras.

We address this lack of faithfulness by constructing the category **Meas**. This is an instance of a general construction [97], but we give the details in full here because of a lack of standard references.

5.8.1 Factorization of Functors

Given a locally small category $\mathcal{C}$, and a functor $F : \mathcal{C} \to \mathcal{D}$, define a relation on elements of $\mathcal{C}(X, Y)$:

$$f \sim g \Leftrightarrow F(f) = F(g).$$

This is an equivalence relation because = is an equivalence relation.

We can start to define a category $\mathcal{C}'$ as follows:

$$\mathrm{Obj}(\mathcal{C}') = \mathrm{Obj}(\mathcal{C})$$
$$\mathcal{C}'(X, Y) = \mathcal{C}(X, Y)/\sim$$

We can define id_X to be $[\mathrm{id}_X]$, and $[f] \circ [g] = [f \circ g]$.

Proposition 5.8.3. $\mathcal{C}'$ *is a category.*

Proof. First we show that composition is well-defined. Suppose $[f'] = [f]$ and $[g'] = [g]$. Then we have $F(f') = F(f)$ and $F(g') = F(g)$, and so

$$F(f' \circ g') = F(f') \circ F(g') = F(f) \circ F(g) = F(f \circ g)$$

so $[f' \circ g'] = [f \circ g]$. Therefore

$$[f'] \circ [g'] = [f' \circ g'] = [f \circ g] = [f] \circ [g]$$

as required.

We move on to showing that composition is associative:

$$\begin{aligned}[f] \circ ([g] \circ [h]) &= [f] \circ [g \circ h] = [f \circ (g \circ h)] = [(f \circ g) \circ h] = [f \circ g] \circ [h] \\ &= ([f] \circ [g]) \circ [h].\end{aligned}$$

Finally, we show that the identity laws hold:

$$[\mathrm{id}_Y] \circ [f] = [\mathrm{id}_Y \circ f] = [f], \text{ and } [f] \circ [\mathrm{id}_X] = [f \circ \mathrm{id}_X] = [f]. \qquad \square$$

Theorem 5.8.4. *Let $\mathcal{C}$ be a locally small category, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. There exists a category $\mathcal{C}'$ and a factorization of F as*

$$\mathcal{C} \xrightarrow{P} \mathcal{C}' \xrightarrow{F'} \mathcal{D} \tag{5.13}$$

with P full and essentially surjective, and F' faithful.

Proof. We just showed that $\mathcal{C}'$ exists and is a category (Proposition 5.8.3). We define $P(X) = X$ and $P(f) = [f]$. The definitions of identity maps and composition in $\mathcal{C}'$ were chosen to make this a functor. It is not only essentially surjective on objects, but in fact surjective on objects. Therefore we will be done with P if we can show that P is full. Given $[f] \in \mathcal{C}'(X, Y)$, we have that $P(f) = [f]$, which shows P is full.

We define $F'(X) = F(X)$ and $F'([f]) = F(f)$. First we must show that this is well-defined. If $[f] = [g]$, then by definition, $F(f) = F(g)$, so it is. It preserves identities:

$$F'([\mathrm{id}_X]) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)} = \mathrm{id}_{F'(X)}.$$

It preserves composition:

$$F'([f] \circ [g]) = F'([f \circ g]) = F(f \circ g) = F(f) \circ F(g) = F'([f]) \circ F'([g]).$$

So F' is a functor.

Finally, we show that F' is faithful. Suppose $F'([f]) = F'([g])$. Then $F(f) = F(g)$ and so $[f] = [g]$. $\square$

Proposition 5.8.5. *If $F : \mathcal{C} \to \mathcal{D}$, being factorized as in* (5.13) *from Theorem 5.8.4 is* *(i) full*

(ii) essentially surjective

then $F' : \mathcal{C}' \to \mathcal{D}$ *is also.*

Proof.

(i) Since F is full, for all $g : F(X) \to F(Y)$, there is an $f : X \to Y$ with $F(f) = g$. Now if we are given $g : F'(X) \to F'(Y)$, we have that g also has the type $F(X) \to F(Y)$ by the definition of F' on objects, so there exists a $g : X \to Y$ like before. Now $[f] \in \mathcal{C}'(X, Y)$, and

$$F'([f]) = F(f) = g$$

proving that F' is full.

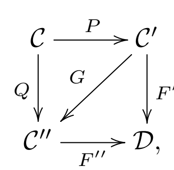
(ii) We have that for each $Y \in \mathrm{Obj}(\mathcal{D})$, there is an $X \in \mathrm{Obj}(\mathcal{C})$ and an isomorphism $i : F(X) \to Y$. Since $X \in \mathrm{Obj}(\mathcal{C}')$ and $F'(X) = F(X)$, we have that for each $Y \in \mathrm{Obj}(\mathcal{D})$ there exists an $X \in \mathrm{Obj}(\mathcal{C}')$ an an isomorphism $i : F'(X) \to Y$. □

We have the following uniqueness theorem for the factorization, where we allow factorizations up to isomorphism, in a sense made precise in the statement of the theorem.

Theorem 5.8.6. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor and let*

$$\mathcal{C} \overset{Q}{\to} \mathcal{C}'' \overset{F''}{\to} \mathcal{D}$$

with Q *full and essentially surjective, and* F'' *faithful, and* $\alpha : F \Rightarrow F'' \circ Q$ *be a natural isomorphism. Then there is a unique equivalence* $G : \mathcal{C}' \to \mathcal{C}''$ *such that the top half of the following diagram commutes:*



and the bottom half commutes up to a natural isomorphism $\alpha' : F' \Rightarrow F'' \circ G$.

Proof. We define G as follows. As the objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$, we define $G(X) = Q(X)$. For a map $[f] \in \mathcal{C}'(X, Y)$, with $f \in \mathcal{C}(X, Y)$, we define $G([f]) = Q(f)$.

- Well-definedness: Suppose $f \sim g$, *i.e.* $F(f) = F(g)$. As α is an isomorphism $F \cong F'' \circ Q$, Lemma 0.4.5 shows that $F''(Q(f)) = F''(Q(g))$. By the faithfulness of F'', we have $Q(f) = Q(g)$, and by the definition of G we have $G([f]) = G([g])$.

- Functoriality:

$$G([\mathrm{id}_X]) = Q(\mathrm{id}_X) = \mathrm{id}_{Q(X)} = \mathrm{id}_{G(X)}$$

  and

$$G([g] \circ [f]) = G([g \circ f]) = Q(g \circ f) = Q(g) \circ Q(f) = G([g]) \circ G([f])$$

- G is essentially surjective: Let $Y \in \mathrm{Obj}(\mathcal{C}'')$. The functor Q is essentially surjective, so there is an $X \in \mathrm{Obj}(\mathcal{C})$ such that $Q(X) \cong Y$. Since $Q(X) = G(X)$, $G(X) \cong Y$.

- G is full: Let $g : G(X) \to G(Y)$ be a morphism in $\mathcal{C}''$. It is therefore a morphism $g : Q(X) \to Q(Y)$. Since Q is full, there is a map $f : X \to Y$ in $\mathcal{C}$ such that $Q(f) = g$. We therefore have that $[f] : X \to Y$ in $\mathcal{C}'$, and $G([f]) = Q(f) = g$.

- G is faithful: Suppose $G([f]) = G([g])$. Then $Q(f) = Q(g)$, so $F''(Q(f)) = F''(Q(g))$, which implies $F(f) = F(g)$ (Lemma 0.4.5). This, by definition, is $f \sim g$ and so $[f] = [g]$.

- $Q = G \circ P$: Observe

$$G(P(X)) = G(X) = Q(X) \text{ and } G(P(f)) = G([f]) = Q(f).$$

- Uniqueness: Suppose there is a $G' : \mathcal{C}' \to \mathcal{C}''$ such that $Q = G' \circ P$ (we do not even require that the other part of the diagram commutes). Then

$$G'(X) = G'(P(X)) = Q(X) = G(X)$$

  and

$$G'([f]) = G'(P(f)) = Q(f) = G(f).$$

  so $G' = G$.

- Definition of α' and naturality: For each $X \in \mathcal{C}$, $\alpha_X : F(X) \to F''(Q(X))$. By definition, $F(X) = F'(P(X))$, and $\mathcal{C}'$ has the same objects as $\mathcal{C}$ and P is the identity map on objects, so $F'(P(X)) = F'(X)$. Furthermore, by the commutativity we have shown earlier, $F''(Q(X)) = F''(G(P(X))) = F''(G(X))$. This means that we can define $\alpha'_X = \alpha_X$ and it has the correct type after all. It is also already known to be a family of isomorphisms, so all we need to show is that it is natural. Let $[f] : X \to Y$ be a morphism in $\mathcal{C}'$. We want to show that

$$\begin{array}{ccc} F'(X) & \xrightarrow{\alpha'_X} & F''(G(X)) \\ {\scriptstyle F'([f])}\downarrow & & \downarrow{\scriptstyle F''(G([f]))} \\ F'(Y) & \xrightarrow[\alpha'_Y]{} & F''(G(Y)) \end{array}$$

commutes. We have $\alpha'_X = \alpha_X$, and likewise for Y, and $F'([f]) = F(f)$, and $F''(G([f])) = F''(Q(f))$. So the diagram is the same as the naturality diagram for α, and therefore commutes. $\square$

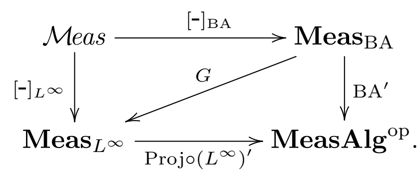
We define **Meas** to be $\mathcal{M}eas'$ for the functor BA, and overload BA to also mean the functor $\mathbf{Meas} \to \mathbf{MeasAlg}^{\mathrm{op}}$. Since BA : $\mathcal{M}eas \to \mathbf{MeasAlg}^{\mathrm{op}}$ is full and essentially surjective (Theorem 5.7.3 and Theorem 5.6.9), by Proposition 5.8.5 BA : $\mathbf{Meas} \to \mathbf{MeasAlg}^{\mathrm{op}}$ is full, faithful and essentially surjective, and hence an equivalence. Thus we have proved

Theorem 5.8.7. *The functor* BA *defines an equivalence*

$$\mathbf{Meas} \simeq \mathbf{MeasAlg}^{\mathrm{op}}.$$

Corollary 5.8.8. L^∞ *defines the same equivalence relation on hom-sets of* $\mathcal{M}eas$ *as* BA*. Therefore* $(L^\infty)'$ *is well defined from* $\mathbf{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$*, and is an equivalence. We also have* $\mathrm{Proj} \circ (L^\infty)' \cong \mathrm{BA}'$.

Proof. We use the isomorphism c' : BA $\Rightarrow$ Proj $\circ\, L^\infty$ from Lemma 5.5.11 in Theorem 5.8.6, factorizing BA as $\mathrm{BA}' \circ [\text{-}]_{\mathrm{BA}}$ and L^∞ as $(L^\infty)' \circ [\text{-}]_{L^\infty}$:



Note that we can apply the theorem because $(L^\infty)'$ is faithful by definition and Proj is an equivalence, so their composite is faithful.

We have that if $f, g : X \to Y$ in $\mathcal{M}eas$, then

$$\begin{aligned} [f]_{L^\infty} = [g]_{L^\infty} &\Leftrightarrow G([f]_{\mathrm{BA}}) = G([g]_{\mathrm{BA}}) && \text{top triangle commutes} \\ &\Leftrightarrow [f]_{\mathrm{BA}} = [g]_{\mathrm{BA}} && G \text{ an equivalence.} \end{aligned}$$

Therefore L^∞ and BA define the same equivalence relation on the hom sets of $\mathcal{M}eas$, so $\mathbf{Meas}_{\mathrm{BA}} = \mathbf{Meas}_{L^\infty} = \mathbf{Meas}$ and G is the identity functor.

We have c'' : $\mathrm{BA}' \Rightarrow \mathrm{Proj} \circ (L^\infty)'$, the natural isomorphism arising from Theorem 5.8.6. As BA' and Proj are equivalences, $(L^\infty)'$ is an equivalence. $\square$

As in the case of BA, we will overload the notation and write $(L^\infty)'$ as L^∞.

5.9 Consequences

We can define a measure space $(1, \mathcal{P}(1), \mu_1)$ with μ_1 the counting measure. This is an object of **Meas**. $L^\infty(1) \cong \mathbb{C}$ as a W*-algebra, so we have

$$\mathbf{Meas}(1, X) \cong \mathbf{CW^*Alg}(L^\infty(X), \mathbb{C}).$$

The right hand side can be considered the "normal spectrum". For each point $x \in X$, we have a map $p_x : 1 \to X$ defined by

$$p_x(*) = x,$$

which is measurable (in terms of measurable spaces) since 1 has the powerset σ-algebra.

Theorem 5.9.1. *p_x is normal iff x is not contained in any nullset, iff $\{x\}$ is not a nullset. For $x, y \in X$, $p_x = p_y$ iff for all $S \in \Sigma$, $x \in S \Leftrightarrow y \in S$. Therefore*

$$\mathbf{Meas}(1, X) \cong \{x \in X \mid \{x\} \notin N(\mu)\}/\sim,$$

where $x \sim y$ is defined to be $\forall S \in \Sigma . x \in S \Leftrightarrow y \in S$.

Proof.

- p_x normal $\Rightarrow$ x not contained in any nullset:

  Suppose for a contradiction that $x \in N$, $N \in \Sigma$, $\mu(N) = 0$. Then $\mu_1(p_x^{-1}(N)) = 0$ since normal maps are null-preserving (Lemma 5.5.2). Since $* \in p^{-1}(N)$, we have a contradiction with $\mu_1(\{*\}) = 1$.

- x contained in no nullset $\Rightarrow$ p_x normal:

  We have already that p_x is a map of measurable spaces. If $N \in \Sigma$ has $\mu(N) = 0$, then $x \notin N$ by assumption, so $p_x^{-1}(N) = \emptyset$, which is a μ_1-nullset, so p_x is null-preserving, or a measurable map. Since X is strictly localizable, given the decomposition $(S_i)_{i \in I}$ of X, there must be some $i \in I$ such that $x \in S_i$, and S_i is μ-finite. Now, $p_x^{-1}(S_i) = 1$, so for any subset $T \subseteq 1$, we have $T \setminus p_x^{-1}(S_i) = \emptyset$. So we have that $\mu_1(T \setminus p_x^{-1}(S_i)) = 0$ and can conclude that p_x is normal by Lemma 5.5.4.

- x not contained in any nullset iff $\{x\}$ is not a nullset:

  If x is contained in a nullset then by completeness $\{x\}$ is a nullset. If x is not contained in a nullset, then $\{x\}$ cannot be a nullset.

- $p_x = p_y$ iff $\forall S \in \Sigma . x \in S \Leftrightarrow y \in S$:

  $p_x = p_y$ in **Meas** is true iff $\mathrm{BA}(p_x) = \mathrm{BA}(p_y)$, by Corollary 5.8.8. Now $\mathrm{BA}(p_x) = \mathrm{BA}(p_y)$ iff for all $S \in \Sigma$, $\mathrm{BA}(p_x)([S]) = \mathrm{BA}(p_y)([S])$. Now

  $$\mathrm{BA}(p_x)([S]) = [p_x^{-1}(S)] = p_x^{-1}(S),$$

  because $\emptyset$ is the only nullset in 1. Likewise, $\mathrm{BA}(p_y)([S]) = p_y^{-1}(S)$. Then $p_x^{-1}(S) = p_y^{-1}(S)$ iff $x \in S \Leftrightarrow y \in S$. □

Corollary 5.9.2. *Every point of (X, Σ, μ) has measure zero iff*

$$\mathbf{CW^*Alg}(L^\infty(X), \mathbb{C}),$$

equivalently

$$\mathbf{MeasAlg}(\mathrm{BA}(X), 2)$$

is empty.

Proof. The conditions on $\mathbf{CW^*Alg}$ and **MeasAlg** are equivalent by Corollary 5.8.8 and the fact that $\mathrm{Proj}(\mathbb{C}) = 2$. We therefore deal only with the first case.

Suppose every point has measure zero and let $\phi : L^\infty(X, \Sigma, \mu) \to \mathbb{C}$ be an element of the normal spectrum. It is a map in $\mathbf{CW^*Alg}$, so there is a map $f : (1, \mathcal{P}(1), \mu_1) \to X$ in $\mathcal{M}eas$ corresponding to it under Corollary 5.8.8. We have that $\mu_1(\{*\}) = 1$, but $\mu(\{f(*)\}) = 0$, so since f is null-preserving, $\mu_1(f^{-1}(f(*))) = 0$, a contradiction.

For the other direction, suppose the normal spectrum is empty. By Theorem 5.9.1, every singleton has measure zero. ☐

Cho [20] defines a functor $\mathcal{K}\ell(\mathcal{D}_\infty) \to \mathbf{CW^*Alg}_{\mathrm{PU}}{}^{\mathrm{op}}$ that is ℓ^∞ on objects and is analogous to those maps defined in Proposition 1.4.2 and Theorem 1.5.1 and is proven to be full and faithful in [20, Corollary 8.1]. The above theorem shows that $L^\infty([0,1], \lambda)$[6] is not in the essential image of this functor, so unlike Theorem 1.5.1, commutativity of W*-algebras is not equivalent to their state spaces being "free", *i.e.* coming from the Kleisli category.

5.10 Relation to Previous Work

The proof of fullness draws substantially on Fremlin's work on measure algebras, in particular [41, Theorem 343B (i), (v)]. Originally the author worked entirely with commutative W*-algebras and with the notion of lifting appropriate to that setting (a map $L^\infty(X) \to \mathcal{L}^\infty(X)$), as defined in [130, Definition III.1.2], and the lifting theorem of [130, Theorem IV.3.5]. It turned out, however, that the proof was too long to include, used the fact that Proj was an equivalence implicitly several times, and did not improve substantially on Fremlin's proof.

Dmitri Pavlov's notion of measurable locale coincides with the underlying Boolean algebra of a localizable measure algebra in Fremlin's sense [41, Corollary 321D] (Theorem 5.6.1). The suggestion in [79, p. 4] that $\mathbf{CW^*Alg}$ is dual to Kornell's **SLM** cannot be made to work. In Kornell's **SLM**, the only measure spaces with the countable chain condition are disjoint unions of a countable set of singletons and a countable set of $[0,1]$s, and thus are Polish spaces. By [42, 4A2P (a) (iii) and Proposition 433A] if $(X, \mathcal{B}o(X), \mu)$ is a locally finite measure on a Polish space, $\mathrm{BA}(X, \mathcal{B}o(X), \mu)$ has countable Maharam type. However for any infinite cardinal κ, $\mathrm{BA}(2^\kappa, \mathcal{B}o(2^\kappa), \mu)$, where μ is the standard coin-flipping probability measure, has Maharam type κ [41, Theorem 331K]. This means that so for any uncountable κ, $L^\infty(2^\kappa)$ is not in the essential image of **SLM** under L^∞, so L^∞ is not an equivalence. Kornell also does not consider that $L^\infty(f)$ might not be normal for a map between discrete measure spaces if a measurable cardinal exists (Counterexample 5.5.8), so if such a cardinal exists, the functor L^∞ is not well-defined from $\mathbf{SLM} \to \mathbf{CW^*Alg}^{\mathrm{op}}$, only to $\mathbf{CC^*Alg}^{\mathrm{op}}$.

We have referred throughout to commutative C*-algebras whose spectra are Stone. These in fact have a characterization not relying on the spectrum, as the commutative AF C*-algebras[53, Lemma A.3].

[6] Where λ is the Lebesgue measure.

Appendix A

Miscellanea

This appendix contains proofs we were unable to find elsewhere.

A.1 Elementary Real Analysis

Lemma A.1.1. *Let* $(P, \leq)$ *be a directed poset, and* $(a_i)_{i \in P}$ *be a monotone net in* $\mathbb{R}$*. Then*

(i) If $\lim_{i \in P} a_i$ *exists then* $\sup_{i \in P} a_i$ *exists and* $\lim_{i \in P} a_i = \sup_{i \in P} a_i$*.*

(ii) If $\sup_{i \in P} a_i$ *exists then* $\lim_{i \in P} a_i$ *exists and it is* $\sup_{i \in P} a_i$*.*

Proof.

(i) First we show that $\lim_{i \in P} a_i$ is an upper bound. Suppose for a contradiction that there is a $j \in P$ such that $a_j > \lim_{i \in P} a_i$. We take $\epsilon = a_j - \lim_{i \in P}$, and by our assumption this is greater than 0. Since (a_i) converges, there exists a $k \in P$ such that for all $m \geq j$ we have $|a_m - \lim_{i \in P} a_i| < \epsilon$. Since P is directed, there is some $m \in P$ such that $m \geq j, k$. Since $m \geq k$, we have $|a_m - \lim_{i \in P} a_i| < \epsilon$. But since $m \geq j$, we have

$$a_m - \lim_{i \in P} a_i \geq a_j - \lim_{i \in P} a_i > 0,$$

so

$$|a_m - \lim_{i \in P} a_i| = a_m - \lim_{i \in P} a_i \geq a_j - \lim_{i \in P} a_i = \epsilon,$$

which is a contradiction. This implies $\lim_{i \in P} a_i$ is an upper bound for $(a_i)_{i \in P}$.

To show it is a least upper bound, let b be an upper bound for (a_i), and suppose for a contradiction that $b < \lim_{i \in P} a_i$. Let $\epsilon = \lim_{i \in P} a_i - b$. By convergence of (a_i) there is a $j \in P$ such that $|\lim_{i\in P} a_i - a_j| < \epsilon$. Since $\lim_{i\in P} a_i$ is an upper bound, we have

$$|\lim_{i\in P} a_i - a_j| = \lim_{i\in P} a_i - a_j < \epsilon = \lim_{i\in P} a_i - b.$$

Therefore $a_j > b$, contradicting the assumption that b is an upper bound.

(ii) Suppose (a_i) has a least upper bound, $\sup_{i\in P} a_i$. We aim to show that (a_i) converges to it, i.e. that for all $\epsilon > 0$, there is some $j \in P$ such that for all $k \geq j$, $|\sup_{i\in P} a_i - a_k| < \epsilon$. So sppose for a contradiction that there is some $\epsilon > 0$ such that for all $j \in P$, there exists a $k \geq j$ such that $|\sup_{i\in P} a_i - a_k| = \sup_{i\in P} a_i - a_k \geq \epsilon$.

Observe that since $a_j \leq a_k$, we have

$$\epsilon \leq \sup_{i\in I} a_i - a_k \leq \sup_{i\in I} a_i - a_j,$$

so it is actually true that for all $j \in P$, $\sup_{i\in P} a_i - a_j \geq \epsilon > 0$. Therefore $\sup_{i\in P} a_i - \frac{\epsilon}{2}$ is a smaller upper bound for (a_i), contradicting the definition of $\sup_{i\in P} a_i$. □

Now we can prove a lemma about monotone nets.

Lemma A.1.2. *Let $(P, \leq)$ be a directed poset, $(a_i)_{i\in P}, (b_i)_{i\in P}$ be monotone nets in $\mathbb{R}$, such that (b_i) converges and $a_i \leq b_i$ for all $i \in P$. Then (a_i) converges.*

Proof. By Lemma A.1.1, $\lim_{i\in P} b_i = \sup_{i\in P} b_i$. Since $a_i \leq b_i \leq \sup_{i\in P} b_i$, (a_i) is bounded above, and therefore has a least upper bound. Therefore $\sup_{i\in I} a_i$ exists, so by A.1.1 again, $\lim_{i\in P} a_i$ exists. □

A.1.1 Sums

We will require some facts about sums of nonnegative reals, where a divergent sum is considered to have the value ∞. What we really need is a discrete, but uncountable analogue of Tonelli's theorem from measure theory. The Tonelli and Fubini theorems require a σ-finiteness hypothesis so we cannot use them directly.[1]

For convenience, when dealing with directed sets, nets and sequences we write $a : D \to P$ when defining the type, and then write a_i for $a(i) \in P$, with $i \in D$.

Lemma A.1.3. *Let P be a poset, and $a : D \to P$ and $b : E \to P$ be directed sets whose suprema exist. If $\forall i \in D.\exists j \in E.a_i \leq b_j$, then $\sup_{i\in D} a_i \leq \sup_{j\in E} b_j$.*

Proof. It suffices to show that $\sup_{j\in E} b_j$ is an upper bound for $(a_i)_{i\in D}$. Since $\forall i \in D.\exists j \in E.a_i \leq b_j$, and $b_j \leq \sup_{j\in E} b_j$, we have that for all $i \in D$, $a_i \leq \sup_{j\in E} b_j$, as required. □

[1]Though perhaps Halmos's definitions in [49] would work as he uses σ-rings.

Lemma A.1.4. *If D is a directed set, and $a, b : D \to \mathbb{R}$ are monotone maps defining directed sets in $\mathbb{R}$, then*

$$\sup_{j \in D} a_j + \sup_{j \in D} b_j = \sup_{j \in D} a_j + b_j$$

Proof. For ease of notation, all suprema will be presumed to be over D in the rest of the proof. Now, since $a_j \leq \sup a_j$ and likewise for b_j, so $a_j + b_j \leq \sup a_j + \sup b_j$, so we have

$$\sup a_j + b_j \leq \sup a_j + \sup b_j.$$

For the other direction, suppose for a contradiction that $\sup a_j + \sup b_j > \sup a_j + b_j$. Then $\sup a_j > \sup(a_j + b_j) - \sup b_j$, so there is some $k \in D$ such that $a_k > \sup(a_j + b_j) - \sup b_j$. Therefore we also have $\sup b_j > \sup(a_j + b_j) - a_k$, so there is an $a_{k'}$ such that $b_{k'} > \sup(a_j + b_j) - a_k$. Since D is directed, there exists $k'' \in D$ such that $k, k \geq k''$, and we have $a_{k''} + b_{k''} > \sup a_j + b_j$, which is a contradiction, finishing the proof. □

Lemma A.1.5. *Let S be a finite set, and let D be a directed set and $a : S \times D \to \mathbb{R}$ be a directed set $D \to \mathbb{R}$ when evaluated at each value of i. Then*

$$\sum_{i \in S} \sup_{j \in D} a_{ij} = \sup_{j \in D} \sum_{i \in S} a_{ij}.$$

Proof. For ease of writing the proof, we replace S with an initial segment of the positive naturals $\{1, \ldots, n\}$. We then proceed by induction. The base case is $n = 1$. Then

$$\sum_{i=1}^{1} \sup_{j \in D} a_{ij} = \sup_{j \in D} a_{1j} = \sup_{j \in D} \sum_{i=1}^{1} a_{ij}.$$

For the inductive step, assume that the statement is true for n. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \sup_{j \in D} a_{ij} &= \sup_{j \in D} a_{n+1,j} + \sum_{i=1}^{n} \sup_{j \in D} a_{ij} \\
&= \sup_{j \in D} a_{n+1,j} + \sup_{j \in D} \sum_{i=1}^{n} a_{ij} && \text{by the inductive hypothesis} \\
&= \sup_{j \in D} \sum_{i=1}^{n+1} a_{ij} && \text{by Lemma A.1.4,}
\end{aligned}$$

which completes the proof. □

The following is the uncountable discrete Tonelli theorem we need.

Lemma A.1.6. *Let X, Y be sets and $a : X \times Y \to [0, \infty)$. Then*

$$\sum_{x \in X} \sum_{y \in Y} a_{xy} = \sum_{(x,y) \in X \times Y} a_{xy} = \sum_{y \in Y} \sum_{x \in X} a_{xy}.$$

Note that as these are sums of positive numbers we allow ∞ as the value of a sum, so convergence of any one of these quantites implies convergence of the other two.

Proof. We prove that

$$\sum_{x\in X}\sum_{y\in Y} a_{xy} = \sum_{(x,y)\in X\times Y} a_{xy},$$

from which the other identity can be deduced by symmetry.

Recall that $A \Subset B$ means A is a finite subset of B. By the definition of summation we are trying to show that

$$\sup_{S\Subset X\times Y} \sum_{(x,y)\in S} a_{xy} = \sup_{S\Subset X}\sum_{x\in S}\sup_{T\Subset Y}\sum_{y\in T} a_{xy}$$

We begin by showing that the left hand side is less than or equal to the right side. We do so by using Lemma A.1.3, reducing to showing

$$\forall S \Subset X\times Y.\exists I \Subset X. \sum_{(x,y)\in S} a_{xy} \leq \sum_{x\in I}\sup_{J\Subset Y}\sum_{y\in J} a_{xy}.$$

To prove this, we take $I = \pi_1(S)$, and we have

$$\sum_{(x,y)\in S} a_{xy} = \sum_{x\in\pi_1(S)}\sum_{y\in\{y|(x,y)\in S\}} a_{xy} \leq \sum_{x\in\pi_1(S)}\sup_{J\Subset Y}\sum_{y\in J} a_{xy}.$$

We therefore only need to show that

$$\sup_{S\Subset X}\sum_{x\in S}\sup_{T\Subset Y}\sum_{y\in T} a_{xy} \leq \sup_{I\Subset X\times Y}\sum_{(x,y)\in I} a_{xy}.$$

We reduce this to showing

$$\forall S \Subset X. \sum_{x\in S}\sup_{T\Subset Y}\sum_{y\in T} a_{xy} \leq \sup_{I\Subset X\times Y}\sum_{(x,y)\in I} a_{xy}.$$

By using Lemma A.1.5, we can reduce this to showing

$$\forall S \Subset X. \sup_{T\Subset Y}\sum_{x\in S}\sum_{y\in T} a_{xy} \leq \sup_{I\Subset X\times Y}\sum_{(x,y)\in I} a_{xy}.$$

By applying Lemma A.1.3, we reduce to showing

$$\forall S \Subset X.\forall T \Subset Y.\exists I \Subset X\times Y. \sum_{x\in S}\sum_{y\in T} a_{xy} \leq \sum_{(x,y)\in I} a_{xy}.$$

We do this by taking $I = S \times Y$. $\square$

A.2 General Topology

It is commonly known that $f : X \to Y$ is continuous iff for all nets $(x_i)_{i\in I}$, $x_i \in X$ we have $\lim_{i\in I} f(x_i) = f(\lim_{i\in I} x_i)$. The following lemmas relate this fact to the subspace topology.

Lemma A.2.1. *Let X be a topological space, $S \subseteq X$ a subspace, $(x_i)_{i\in I}$ a net with $x_i \in S$ for all $i \in I$, and $x \in S$. Then $\lim_{i\in I} x_i = x$ in X's topology iff $\lim_{i\in I} x_i = x$ in S's subspace topology.*

Proof.

- $x_i \to x$ in S implies $x_i \to x$ in X:

  Suppose $x_i \to x$ in S, *i.e.* for all $U \in \mathcal{O}(S)$ such that $x \in U$, there exists $j_U \in I$ such that for all $i \geq j_U$ we have $x_i \in U$. Then if $U \in \mathcal{O}(X)$ and $x \in U$, $U \cap S \in \mathcal{O}(S)$ so we take $j_{U\cap S}$ to prove the convergence of x_i to x, as $U \cap S \subseteq U$.

- $x_i \to x$ in X implies $x_i \to x$ in S:

  Now suppose $x_i \to x$ in X, and define j_U for each $U \in \mathcal{O}(X)$ such that $x \in U$ as before. If $U \in \mathcal{O}(S)$, we have $U = U' \cap S$ for $U' \in \mathcal{O}(X)$. So if $x \in U$, we have $x \in U'$ and so there exists a $j_{U'}$ such that for all $i \geq j_{U'}$ we have $x_i \in U'$. Since $x_i \in S$, we also have that $x_i \in U' \cap S = U$, so $x_i \to x$ in S too. □

Corollary A.2.2. *Let $f : X \to Y$ be a function, where X and Y are topological spaces. Let $S \subseteq X$. Then $f|_S$ is continuous iff for all nets $(x_i)_{i\in I}$ with $x_i \in S$ that converge to a point $x \in S$, we have $\lim_{i\in I} f(x_i) = f(x)$.*

Proof. By [75, Chapter 3, Theorem 1] the map $f|_S$ is continuous iff for all nets $(x_i)_{i\in I}$ in S converging to a point x in S, $(f(x_i))_{i\in I}$ converges to $f(x)$ in Y. We then use Lemma A.2.1 to deduce that this is so iff for all $(x_i)_{i\in I}$ converging to $x \in S$ in the topology of X, $(f(x_i))_{i\in I}$ converges to $f(x)$ in the topology of Y. □

Let $U : \mathbf{CHaus} \to \mathbf{Set}$ be the forgetful functor.

Proposition A.2.3. *Every continuous function on a compact Hausdorff space is bounded. Therefore we have an inclusion $\iota_X : C(X) \to \ell^\infty(U(X))$. This is a natural transformation $\iota : C \Rightarrow \ell^\infty U$. This is so whether we take C and ℓ^∞ to be C^*-algebras of $\mathbb{C}$-valued functions or Banach order-unit spaces of $\mathbb{R}$-valued functions.*

Proof. The reason that every continuous function on a compact Hausdorff space is bounded is that the image of a compact set is compact, and every compact subset of $\mathbb{R}$ (respectively, $\mathbb{C}$) is bounded by the Heine-Borel theorem. The map ι_X is a map in **BOUS** because the vector space structure, positive cone and order unit are exactly the same for $C(X)$ and $\ell^\infty(U(X))$, or it is a map in

C*Alg because the multiplication, involution and vector space structure are the same. Thus we only need to show that this forms a natural transformation. Let $f : X \to Y$ be a continuous map between compact Hausdorff spaces. We want to show that

$$\begin{array}{ccc} C(Y) & \xrightarrow{\iota_Y} & \ell^\infty(U(Y)) \\ {\scriptstyle C(f)}\downarrow & & \downarrow{\scriptstyle \ell^\infty(U(f))} \\ C(X) & \xrightarrow[\iota_X]{} & \ell^\infty(U(X)) \end{array}$$

commutes.

So let $b \in C(Y)$, and for the lower left way:

$$\iota_X(C(f)(b)) = b \circ f,$$

while for the upper right way:

$$\ell^\infty(U(f))(\iota_Y(b)) = b \circ f$$

hence the diagram commutes. □

A.3 Absolutely Convex Sets

Lemma A.3.1. *A set is absolutely convex iff it is balanced, convex and nonempty, and in either case contains* 0*.*

Proof. If E is a real vector space and $S \subseteq E$ an absolutely convex set, we can see that it is convex because each convex combination is also an absolutely convex combination. It is balanced because $-x$ is an absolutely convex combination of x. It is non-empty because we can take the empty absolutely convex combination to get $0 \in S$.

On the other hand, let S be balanced, convex and non-empty. We know there is an $x \in S$, so $-x \in S$ by balancedness, and $\frac{1}{2}x + \frac{1}{2}(-x) \in S$ by convexity. Therefore $0 \in S$.

Now, let $\sum_{i=1}^{n} \alpha_i x_i$ be an absolutely convex combination of elements of S. Define $\beta_i = |\alpha_i|$ for $i \in \{1 \dots n\}$ and

$$\beta_{n+1} = 1 - \sum_{i=1}^{n} |\alpha_i|,$$

which is an element of $[0, 1]$. We then see that $\sum_{i=1}^{n+1} \beta_i = 1$. We define $y_i = \operatorname{sgn}(\alpha_i) x_i$ for $i \in \{1 \dots n\}$ and $y_{n+1} = 0$. Then $\sum_{i=1}^{n+1} \beta_i y_i$ is a convex combination of elements of S, and so is in S by its convexity. All we need is

$$\sum_{i=1}^{n+1} \beta_i y_i = \sum_{i=1}^{n} |\alpha_i| \operatorname{sgn}(\alpha_i) x_i + \beta_{n+1} 0 = \sum_{i=1}^{n} \alpha_i x_i$$

to show that S is absolutely convex. □

A.4 Effect Modules

Lemma A.4.1. *Let A be an effect module. If $a, b \in A$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, then $\alpha x \perp \beta y$ in A. In particular, A is closed under convex combinations (when $\beta = 1 - \alpha$). Any effect module morphism $f : A \to B$ is an affine map with respect to these convex combinations.*

Proof. We have

$$\alpha u = \alpha(a \ovee a^{\perp}) = \alpha a \ovee \alpha a^{\perp}$$
$$\beta u = \beta(b \ovee b^{\perp}) = \beta b \ovee \beta b^{\perp}.$$

Define $\gamma = \alpha + \beta \in [0,1]$. Since $\alpha u \ovee \beta u = \gamma u$, we have that $\alpha a \ovee \alpha a^{\perp} \perp \beta b \ovee \beta b^{\perp}$, and their sum is γu. Using associativity, we see that $((\alpha a \ovee \alpha a^{\perp}) \ovee \beta b) \perp \beta b^{\perp}$. We then use commutativity and associativity as follows

$$\begin{aligned} \alpha a \ovee \alpha a^{\perp} \perp \beta b &\Leftrightarrow \alpha a^{\perp} \ovee \alpha a \perp \beta b \\ &\Rightarrow \alpha a \perp \beta b. \end{aligned}$$

To see that an effect module morphism $f : A \to B$ is affine, observe that

$$f(\alpha x \ovee (1-\alpha)y) = \alpha f(x) \ovee (1-\alpha) f(y)$$

by the definition of an effect module morphism. □

Lemma A.4.2. *In any effect module, if $\alpha \in [0,1]$, we have $\alpha \cdot 0 = 0$.*

Proof. We have that $\alpha \cdot 0 \ovee (1-\alpha) \cdot 0 = 0$ by the effect module axioms. Therefore $\alpha \cdot 0 \ovee (1-\alpha) \cdot 0 \perp 1$. By associativity and commutativity we have therefore that $\alpha \cdot 0 \perp 1$, and so $\alpha \cdot 0 = 0$ by the effect algebra axioms. □

A.5 Order-Unit Spaces

Here we show that the two possible definitions of strong order-unit coincide.

Lemma A.5.1. *For an ordered vector space (E, E_+) with chosen element $u \in E_+$, the following two statements are equivalent:*

(i) $E = \bigcup_{n \in \mathbb{N}} [-nu, nu]$

(ii) $E_+ = \bigcup_{n \in \mathbb{N}} [0, nu]$ *and E_+ is generating (or E is directed).*

Proof.

- (i) $\Rightarrow$ (ii):

  First we show that E_+ is generating. Given $x \in E$, we have the existence of some $n \in \mathbb{N}$ such that $x \in [-nu, nu]$. Therefore $nu - x \in E_+$. Since $nu \in E_+$ too, we can define $x_+ = nu$ and $x_- = nu - x$, and then $x = x_+ - x_-$.

Now we show the condition on positive elements. The interval $[0, \nu] \subseteq E_+$, so $\bigcup_{n \in \mathbb{N}} [0, nu] \subseteq E_+$. For the other inclusion, suppose that $x \in E_+$. We have that $x \in [-nu, nu]$ for some n by assumption, and then we simply apply the fact that $x \geq 0$ to deduce that $x \in [0, nu]$.

- (ii) $\Rightarrow$ (i):

  Let $x \in E$. Since E_+ is generating, $x = x_+ - x_-$ for $x_+, x_- \in E_+$. Since this equation can be rearranged as $x_+ - x = x_-$, showing $x_+ \geq x$, and $x - (-x_-) = x + x_- = x_+$, showing $x_- \leq x$. Using the alternative definition of an order unit, there are $m, n \in \mathbb{N}$ such that $x_+ \in [0, mu]$ and $x_- \in [0, nu]$. We have that $-x_- \in [-nu, 0]$. We can then apply transitivity of the order and deduce $x \in [-nu, mu]$. If we take p to be whichever of $\{n, m\}$ is larger, we have $x \in [-pu, pu]$, as required. □

Recall that $(\{0\}, \{0\}, 0)$ is an order-unit space by our definition.

This is the reason for the apparent contradiction between the following Proposition 1.2.8 and [6, Lemma 1.15].

Lemma A.5.2. *If (A, A_+, u) is an order-unit space, the following three properties are equivalent.*

(i) $A \neq 0$

(ii) $u \neq 0$

(iii) $\|u\| = 1$

Proof.

- (i) $\Rightarrow$ (ii): Suppose that $A \neq 0$. Then there is some $a \in A$, and $a = a_+ - a_-$, so there is some positive element $0 \neq a' \in F$. If u were 0, we would have $\alpha u = u$ for all $\alpha \in [0, \infty)$, contradicting its being a strong unit. Therefore $u \neq 0$.

- (ii) $\Rightarrow$ (iii):

  By definition, $u \in [-u, u]$, so $\|u\| \leq 1$. Suppose that $u \in [-\alpha u, \alpha u]$ for $0 \leq \alpha < 1$. Then $u \leq \alpha u$, *i.e.* $(\alpha - 1)u \in A_+$. Since u is positive and $\alpha - 1 < 0$, we have that $-u \in A_+$ and so $u = 0$. By contraposition, if $u \neq 0$, $\|u\| \geq 1$, and so $\|u\| = 1$.

- (iii) $\Rightarrow$ (i):

  If $A = 0$, the only element is 0, and $\|0\| = 0$. So the existence of any element of nonzero norm implies $A \neq 0$. □

There is another characterization of when an ordered vector space with strong unit is an order-unit space, *i.e.* when it is archimedean.

Lemma A.5.3. *Let (A, A_+, u) be an ordered vector space with strong order unit u. Let $\|\text{-}\| = \|\text{-}\|_{[-u,u]}$ be the Minkowski seminorm. Then A_+ is $\|\text{-}\|$-closed iff (A, A_+, u) is archimedean. Furthermore, if (A, A_+, u) is Archimedean, the seminorm $\|\text{-}\|_{[-u,u]}$ is a norm with unit ball $[-u, u]$.*

Proof. We first show that if (A, A_+, u) is archimedean, then A_+ is $\|\text{-}\|$-closed. Let $x \in \mathrm{cl}\,(A_+)$. This means that for all $n \in \mathbb{N}$, we have that there exists a $y \in A_+$ such that $\|x - y\| < \frac{1}{n}$. We know that $\mathrm{Ball}(A) \subseteq [-2u, 2u]$ (Lemma 0.1.6), so for all $n \in \mathbb{N}$ we have $x - y \in \frac{1}{n}[-2u, 2u]$. By redefining n to $2n$ and using Lemma 0.2.2 we have that for all $n \in \mathbb{N}$ there exists a $y \in A_+$ such that $y \in [x - \frac{1}{n}u, x + \frac{1}{n}u]$. This implies that $x + \frac{1}{n}u - y \in A_+$ and $y \in A_+$, so $x + \frac{1}{n}u \in A_+$. Therefore $-x \leq \frac{1}{n}u$ for all $n \in \mathbb{N}$, so by archimedeanness we have $-x \in -A_+$, so $x \in A_+$. Therefore $\mathrm{cl}\,(A_+) = A_+$, *i.e.* A_+ is closed.

We now show that if the cone is closed, (A, A_+, u) is archimedean. So suppose that for all $n \in \mathbb{N}$, we have $a \leq \frac{1}{n}u$, equivalently $\frac{1}{n}u - a \in A_+$. If we show that $\left(\frac{1}{n}u - a\right)_n \to -a$, we can conclude that $-a \in A_+$ because A_+ is closed, and therefore $a \in -A_+$. So let $\epsilon \in \mathbb{R}_{>0}$, and take $n = \lceil \epsilon^{-1} + 1 \rceil$. We then aim to show that for all $i \geq n$ we have $\|(\frac{1}{i}u - a) - (-a)\| < \epsilon$, or equivalently $\|\frac{1}{i}u\| < \epsilon$. Since $\frac{1}{i}u \in \frac{1}{i}[-u, u]$, we have $\|\frac{1}{i}u\| \leq i$.

Now, since $i \geq n$, we have $\frac{1}{i} \leq \frac{1}{n}$. So we have

$$\left\|\frac{1}{i}u\right\| \leq \frac{1}{n} = \frac{1}{\lceil \epsilon^{-1} + 1 \rceil}.$$

We have

$$n = \lceil \epsilon^{-1} + 1 \rceil \geq \epsilon^{-1} + 1 > \epsilon^{-1},$$

so

$$\left\|\frac{1}{i}u\right\| \leq \frac{1}{n} < \epsilon.$$

As explained earlier, the closedness of A_+ implies $-a \in A_+$ and so $a \in -A_+$.

Finally, we want to show that $\|\text{-}\|_{[-u,u]}$ is a norm and $\mathrm{Ball}(\|\text{-}\|_{[-u,u]}) = [-u, u]$ if (A, A_+, u) is archimedean. We show $\|\text{-}\|_{[-u,u]}$ is a norm by showing $[-u, u]$ contains no line through the origin and using Lemma 0.1.5. Suppose for a contradiction that $x \in [-u, u]$ generates a line through the origin contained in $[-u, u]$. We therefore have $nx \leq u$ for all $n \in \mathbb{N}_{>0}$, so $x \leq \frac{1}{n}u$ too, and so $x \in -A_+$. However, this same argument can be applied with $-nx \leq u$ to show $-x \in -A_+$, and since A_+ is assume to be a cone, $x = 0$. This contradicts x generating a line.

To show $[-u, u]$ is the unit ball, by Lemma 0.1.6, the interval $[-u, u] \subseteq \mathrm{Ball}(\|\text{-}\|_{[-u,u]})$, so we show the opposite inclusion. By Lemmas 0.1.6 and 0.2.2

$$\mathrm{Ball}(E) = \bigcap_{1<\alpha<\infty} \alpha[-u, u] = \bigcap_{1<\alpha<\infty} [-\alpha u, \alpha u].$$

If $x \in \mathrm{Ball}(E)$, we therefore have $x - u \leq \frac{1}{n}u$ for all $n \in \mathbb{N}_{>0}$, so $x - u \in -E_+$, *i.e.* $x \leq u$. By the same argument applied to $-x$ we get $-u \leq x$, so we have $x \in [-u, u]$, as required. □

We will require this fact about positive linear functionals on order-unit spaces more than once.

Lemma A.5.4. *Let* (A, A_+, u) *be an order-unit space and* $\phi : A \to \mathbb{R}$ *a positive linear functional. If* $\phi(u) = 0$ *then* $\phi = 0$.

Proof. By linearity, $\phi(\alpha u) = \alpha \cdot 0 = 0$ for all $\alpha \in \mathbb{R}$. For each $a \in A$, there exists $\alpha \in \mathbb{R}_{>0}$ such that $-\alpha u \leq a \leq \alpha u$. As positive linear maps are monotone, we have

$$0 = \phi(-\alpha u) \leq \phi(a) \leq \phi(\alpha u) = 0,$$

$\phi(a) = 0$ for all $a \in A$, *i.e.* $\phi = 0$. □

A.6 Asimow's Example

Here we give Asimow's example of a Banach base-norm space whose unit ball is not radially compact. This example was originally published in [36], without proof.

Recall the Banach space c_0 of sequences of real numbers converging to 0 [32, IV.2 Example 7][24, p. 65], with pointwise vector space operations and its norm being the usual supremum norm. Recall also that there is a bilinear pairing between c_0 and ℓ^1 defined by

$$\langle a, b \rangle = \sum_{i=0}^{\infty} a_i b_i,$$

where $(a_i) \in c_0$ and $(b_i) \in \ell^1$, which defines an isomorphism $\ell^1 \cong c_0^*$[24, Example III.5.8].

The underlying space of Asimow's example is

$$E = \left\{ (x_i) \in c_0 \mid x_1 + x_2 = \sum_{i=3}^{\infty} \frac{x_i}{2^{i-2}} \right\}.$$

To avoid confusion, we stress at this point that the sequences are considered to start at x_0, not x_1. If we define $\phi = (0, -1, -1, \frac{1}{2}, \ldots, \frac{1}{2^{i-2}}, \ldots)$, we can see that $\phi \in \ell^1$, the sum of the absolute values of its entries being 3, and so $\langle \text{-}, \phi \rangle : c_0 \to \mathbb{R}$ is a continuous linear functional. As $E = \langle \text{-}, \phi \rangle^{-1}(0)$, we have shown that E is a closed subset of c_0, and therefore a Banach space in the usual norm of c_0.

Following Asimow, we define $K \subseteq E$ as

$$K = \{(x_i) \in E \mid x_0 = 1 = \|(x_i)\|_{c_0} \text{ and } \forall i \in \mathbb{N}. x_i \geq 0\},$$

and we define E_+ to be the wedge generated by K, *i.e.*

$$E_+ = \{\alpha x \mid \alpha \in \mathbb{R}_{\geq 0} \text{ and } x \in K\}.$$

Take $F = E_+ - E_+$, or equivalently to be the span of K. We can define $\tau : E \to \mathbb{R}$ as

$$\tau((x_i)) = x_0$$

Proposition A.6.1. *The subspace $F = E$, and (E, E_+, τ) is a Banach base-norm space with base K.*

Proof. First we describe all the steps in the proof. We first show that K is convex and E_+ is a proper cone. Then we show that $\tau : F \to \mathbb{R}$ is positive and not zero, and that $E_+ \cap \tau^{-1}(1) = K$. Then we show that $\text{absco}(K) \subseteq \text{Ball}(\|\text{-}\|_{c_0})$, which shows that $\text{absco}(K)$ is radially bounded and that F is a pre-base-norm space. We then show that $\frac{1}{6}\text{Ball}(\|\text{-}\|_{c_0}) \subseteq \text{absco}(K)$, where $\text{Ball}(\|\text{-}\|_{c_0})$ is the unit ball of $\|\text{-}\|_{c_0}$ restricted to E, and therefore $E = F$ and so (E, E_+, τ) is a Banach pre-base-norm space. We then show that K is complete in the c_0-norm, and so therefore is E_+ by Lemma 2.2.14, so E_+ is closed and (E, E_+, τ) is a Banach base-norm space.

- K is convex:

  Let $(x_i), (y_i) \in K$ and $\alpha \in [0, 1]$. We have $\alpha x_0 + (1-\alpha) y_0 = \alpha + (1-\alpha) = 1$. Since x_i and $y_i \geq 0$, we have $\alpha x_i + (1-\alpha) y_i \geq 0$. Then using the subadditivity of the norm

  $$\|\alpha(x_i) + (1-\alpha)(y_i)\|_{c_0} \leq \alpha\|(x_i)\| + (1-\alpha)\|(y_i)\| = \alpha + 1 - \alpha = 1,$$

  and because $\alpha x_0 + (1-\alpha) y_0 = 1$, we have $\|\alpha(x_i) + (1-\alpha)(y_i)\| \geq 1$, so we have shown $\alpha(x_i) + (1-\alpha)(y_i) \in K$.

- E_+ is a proper cone:

  We see immediately that E_+ is closed under positive scalar multiplication. If $\alpha x, \beta y \in E_+$ (*i.e.* $x, y \in K$, $\alpha, \beta \in \mathbb{R}_{\geq 0}$), then

  $$\alpha x + \beta y = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y\right).$$

  We have $\alpha + \beta \in \mathbb{R}_{\geq 0}$, and the rest is a convex combination of elements of K, so is an element of K. We have shown that E_+ is a wedge. Now, suppose that $\alpha(x_i) = -\beta(y_i)$. Then in particular, $\alpha x_0 = -\beta x_0$ and so $\alpha = -\beta$. As both of these are in $\mathbb{R}_{\geq 0}$, we have $\alpha = \beta = 0$ and so $E_+ \cap -E_+ = \{0\}$, and therefore E_+ is a cone.

- τ is linear, positive and nonzero:

  The map τ is linear because the vector space operations on E are pointwise. If $\alpha(x_i) \in E_+$, then $\tau(\alpha(x_i)) = \alpha x_0 = \alpha \geq 0$, so τ is positive. If we take any element of $(x_i) \in K$, such as $(x_i) = (1, 0, \ldots)$, then $\tau((x_i)) = 1$, so τ is nonzero.

- $E_+ \cap \tau^{-1}(1) = K$:

  We already saw that if $x \in K$, we have $\tau(x) = 1$, and by definition $x \in E_+$, so we have $K \subseteq E_+ \cap \tau^{-1}(1)$. For the other inclusion, let $\alpha(x_i) \in E_+$. Then $\tau(\alpha(x_i)) = \alpha x_0 = \alpha$, so if $\tau(\alpha(x_i)) = 1$, we have $\alpha(x_i) = (x_i)$ and therefore $\alpha(x_i) \in K$.

- $\text{absco}(K) \subseteq \text{Ball}(\|\text{-}\|_{c_0})$ and $(F, E_+, \tau) \in \mathbf{PreBNS}$:

  By definition of K, we have $K \subseteq \text{Ball}(\|\text{-}\|_{c_0})$, and since $\text{Ball}(\|\text{-}\|_{c_0})$ is a absolutely convex, $\text{absco}(K) \subseteq \text{Ball}(\|\text{-}\|_{c_0})$. By Lemma 0.1.5, $\text{Ball}(\|\text{-}\|_{c_0})$ is radially bounded, so $\text{absco}(K)$ is radially bounded. We have therefore shown (F, E_+, τ) is a pre-base-norm space.

- $\frac{1}{6}\text{Ball}(\|\text{-}\|_{c_0}) \subseteq \text{absco}(K)$:

  Recall that $\text{Ball}(\|\text{-}\|_{c_0})$ is taken to mean the unit ball of the c_0-norm restricted to E. Let $(x_i) \in \frac{1}{6}\text{Ball}(\|\text{-}\|_{c_0})$, *i.e.* $(x_i) \in E$ and $\|x_i\| \leq \frac{1}{6}$. It follows that for all $i \in \mathbb{N}$, we have $-\frac{1}{6} \leq x_i \leq \frac{1}{6}$. We want to produce $(y_i), (z_i) \in K$ and $\alpha \in [0,1]$ such that for all $i \in \mathbb{N}$, $\alpha y_i - (1-\alpha)z_i = x_i$. We know that no matter what, we must take $y_0 = z_0 = 1$, and therefore

  $$\alpha = \frac{x_0 + 1}{2},$$

  and therefore the bounds on x_0 give

  $$\frac{5}{12} \leq \alpha \leq \frac{7}{12}$$

  How we define y_i and z_i for the other coordinates depends on a case split.

  - If $x_2 \geq 0$:

    For reasons that will become clear later, we will want to define y_i so that

    $$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i \geq \frac{x_2}{\alpha},$$

    so we want to make y_i as large as possible, while still remaining within c_0. To do this, we first define sequences $(\upsilon_i)_{i\geq 3}, (\zeta_i)_{i\geq 3}$, taking values in $[0,1]$, such that $\alpha\upsilon_i - (1-\alpha)\zeta_i = x_i$ and $\sum_{i=3}^{\infty} 2^{-(i-2)}\upsilon_i > \frac{x_2}{\alpha}$. We will ultimately define y_i and z_i using a truncation of these sequences. We define them as follows:

    $$\upsilon_i = 1 \wedge \left(\frac{1-\alpha+x_i}{\alpha}\right) \qquad \zeta_i = 1 \wedge \left(\frac{\alpha - x_i}{1-\alpha}\right),$$

    where $\wedge$ is the usual lattice operation on $\mathbb{R}$, *i.e.* the greatest lower bound, which is the minimum in this case.

    * $\upsilon_i, \zeta_i \in [0,1]$: Since $1 \wedge x \leq 1$ for all x, we have $\upsilon_i, \zeta_i \leq 1$. To show nonnegativity, we only need to show that $\frac{1-\alpha+x_i}{\alpha} \geq 0$ and $\frac{\alpha - x_i}{1-\alpha} \geq 0$, because we already know $1 \geq 0$. As we have $1-\alpha \geq \frac{5}{12}$ and $x_i \geq -\frac{1}{6}$, so $1-\alpha+x_i \geq \frac{3}{12} \geq 0$. Therefore $\frac{1-\alpha+x_i}{\alpha} \geq 0$. We have the same inequalities for α and $-x_i$, so we have $\frac{\alpha-x_i}{1-\alpha} \geq 0$ also.

* $\alpha\upsilon_i - (1-\alpha)\zeta_i = x_i$:
  If $\frac{1-\alpha+x_i}{\alpha} \leq 1$, then $\frac{\alpha-x_i}{1-\alpha} \geq 1$ and vice-versa, so there are two cases. The first is that $\upsilon_i = \frac{1-\alpha+x_i}{\alpha}$ and $\zeta_i = 1$, so

  $$\alpha\frac{1-\alpha+x_i}{\alpha} - (1-\alpha)\cdot 1 = x_i.$$

  The other case is that $\upsilon_i = 1$ and $\zeta_i = \frac{\alpha-x_i}{1-\alpha}$. Then

  $$\alpha \cdot 1 - (1-\alpha)\frac{\alpha - x_i}{1-\alpha} = x_i,$$

  so in either case, we are done.
* $\sum_{i=3}^{\infty} 2^{-(i-2)}\upsilon_i > \frac{x_2}{\alpha}$:
  We want to show

  $$\sum_{i=3}^{\infty} 2^{-(i-2)}\upsilon_i = \sum_{i=3}^{\infty} 2^{-(i-2)}\left(1 \wedge \left(\frac{1-\alpha+x_i}{\alpha}\right)\right) > \frac{x_2}{\alpha},$$

  and as $\alpha \geq 0$, this is equivalent to

  $$\sum_{i=3}^{\infty} 2^{-(i-2)}(\alpha \wedge (1-\alpha+x_i)) > x_2.$$

  We start the argument as follows. We have $\alpha \geq \frac{5}{12}$, and $1-\alpha+x_i \geq \frac{5}{12} + -\frac{1}{6} = \frac{3}{12}$, so $\alpha \wedge (1-\alpha+x_i) \geq \frac{3}{12}$. Therefore

  $$\sum_{i=3}^{\infty} 2^{-(i-2)}(\alpha \wedge (1-\alpha+x_i)) \geq \sum_{i=3}^{\infty} 2^{-(i-2)}\frac{3}{12} = \frac{3}{12} > \frac{1}{6} \geq x_2.$$

Convergence of the sum implies there exists an $N \in \mathbb{N}$ such that $\sum_{i=3}^{N} 2^{-(i-2)}\upsilon_i > \frac{x_2}{\alpha}$. We define y_i and z_i as follows:

$$y_i = \begin{cases} \upsilon_i & \text{if } i \leq N \\ \frac{x_i}{\alpha} & \text{if } i > N \text{ and } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$z_i = \begin{cases} \zeta_i & \text{if } i \leq N \\ -\frac{x_i}{1-\alpha} & \text{if } i > N \text{ and } x_i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

By combining the inequalities on x_i and α, we have $\frac{x_i}{\alpha} \leq \frac{2}{5}$ and the $x_i \geq 0$ in the case that $\frac{x_i}{\alpha}$ occurs ensures that $0 \leq y_i \leq 1$. A similar argument using the inequalities for $1-\alpha$ ensures $0 \leq z_i \leq 1$. We therefore have that y_i, z_i are in $[0,1]$ and that the sequences converge to 0 because they are eventually equal to a subsequence of a multiple of x_i, which converges to 0. We can now define

$$y_1 = \sum_{i=3}^{\infty} y_i - \frac{x_2}{\alpha} \qquad y_2 = \frac{x_2}{\alpha}$$

$$z_1 = \sum_{i=3}^{\infty} z_i \qquad z_2 = 0.$$

We first show that $(y_i), (z_i) \in K$. The condition that $y_0, z_0 = 1$ is satisfied by definition. We have already shown that for $i \geq 3$ we have $y_i, z_i \in [0, 1]$ and y_i and z_i converge to 0.

We arranged that $y_1 \geq 0$ by taking N sufficiently large that $\sum_{i=3}^{N} 2^{-(i-2)} \upsilon_i > \frac{x_2}{\alpha}$, and $\sum_{i=3}^{\infty} y_i \geq \sum_{i=3}^{N} 2^{-(i-2)} \upsilon_i$ because the rest of the y_i for $i > N$ are nonnegative. We also have

$$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i \leq \sum_{i=3}^{\infty} 2^{-(i-2)} \cdot 1 = 1$$

and so $y_1 \leq 1$ because $x_2 \geq 0$. Then $x_2 \geq 0$ also implies $y_2 \geq 0$, and as $x_2 \leq \frac{1}{6}$, we have

$$y_2 = \frac{x_2}{\alpha} \leq \frac{1}{6} \cdot \frac{12}{5} = \frac{2}{5} \leq 1.$$

Then $z_1 \in [0, 1]$ because it is a sum of nonnegative numbers and

$$z_1 = \sum_{i=3}^{\infty} 2^{-(i-2)} z_i \leq \sum_{i=3}^{\infty} 2^{-(i-2)} \cdot 1 = 1.$$

We have $z_2 \in [0, 1]$ because it is 0. We have therefore shown that $y_i, z_i \geq 0$ and $\|(y_i)\| = \|(z_i)\| = 1$. Therefore, to complete the proof that $(x_i), (y_i) \in K$, we only need to show that $y_1 + y_2 = \sum_{i=3}^{\infty} 2^{-(i-2)} y_i$ and likewise for (z_i), and this follows trivially from the definitions in each case.

We can now show $\alpha y_i - (1 - \alpha) z_i = x_i$. For $3 \leq i \leq N$, we have already shown $\alpha \upsilon_i - (1 - \alpha) \zeta_i = x_i$. For $i > N$ and $x_i \geq 0$, we have $\alpha \frac{x_i}{\alpha} - (1 - \alpha) \cdot 0 = x_i$. For $i > N$ and $x_i \leq 0$, we have $\alpha 0 - (1 - \alpha) \frac{-x_i}{1 - \alpha} = x_i$ too. For $i = 0$ we have $\alpha \cdot 1 - (1 - \alpha) \cdot 1 = 2\alpha - 1 = x_0$. For $i = 2$ we have $\alpha \frac{x_2}{\alpha} - (1 - \alpha) \cdot 0 = x_2$. Finally, for $i = 1$ we have

$$\begin{aligned}
&\alpha \left(\sum_{i=3}^{\infty} 2^{-(i-2)} y_i - \frac{x_2}{\alpha} \right) - (1 - \alpha) \sum_{i=3}^{\infty} 2^{-(i-2)} z_i \\
&= -x_2 + \sum_{i=3}^{\infty} 2^{-(i-2)} (\alpha y_i - (1 - \alpha) z_i) \\
&= -x_2 + \sum_{i=3}^{\infty} 2^{-(i-2)} x_i = x_1
\end{aligned}$$

because $(x_i) \in E$.

- If $x_2 \leq 0$:
  Define $x'_i = -x_i$ and apply the previous case to obtain $\alpha' \in [0,1]$ and $(y'_i), (z'_i) \in K$ such that $\alpha' y'_i - (1-\alpha')z'_i = x'_i$. Then define $\alpha = (1-\alpha')$, $y_i = z'_i$ and $z_i = y'_i$ and we have $x_i = \alpha y_i - (1-\alpha)z_i$.

- K complete in $\|\text{-}\|_{c_0}$:

  Let x_i be a Cauchy sequence of elements of K in the c_0-norm. As E is complete, there exists some $y \in E$ such that $x_i \to y$. In particular, we have that for all $j \in \mathbb{N}$, $x_{ij} \to y_j$ in the usual topology of real numbers. As each $x_{i0} = 1$, we have $y_0 = 1$. For each $j \in \mathbb{N}$, we have $x_{ij} \geq 0$, so $y_j \geq 0$. Any norm is continuous in the topology it defines, so we also have $\|y\| = \lim_{i\to\infty} \|x_i\| = 1$, and therefore $y \in K$, and so K is complete.

- $E = F$ and (E, E_+, τ) a Banach base-norm space:

  By definition, $F \subseteq E$. If $x \in E$, then

  $$x \in \|x\|_{c_0}\mathrm{Ball}(\|\text{-}\|_{c_0}) \subseteq \|x\|_{c_0} 6\mathrm{absco}(K) \subseteq F.$$

  Therefore $F = E$, so (E, E_+, τ) is a Banach pre-base-norm space. By Lemma 2.2.14, E_+ is complete in the c_0-norm, which is equivalent to the base-norm, therefore E_+ is closed in the base-norm, and so (E, E_+, τ) is a Banach base-norm space. □

Now define $x = (0, \frac{1}{2}, -\frac{1}{2}, 0, \ldots) \in E$.

Counterexample A.6.2. *The point $x \notin \mathrm{absco}(K)$, but for all $\alpha \in [0,1)$, $\alpha x \in \mathrm{absco}(K)$. Therefore (E, E_+, τ) is a Banach base-norm space that is not radially compact.*

Proof. The first part of the proof is to show that $x \notin \mathrm{absco}(K)$. Recall that, by Lemma 0.1.1, we want to show that there can be no $\alpha \in [0,1]$, $(y_i), (z_i) \in K$ such that $\alpha y_i - (1-\alpha)z_i = x_i$. So assume for a contradiction that such a decomposition of x exists. Then

$$0 = x_0 = \alpha y_0 - (1-\alpha)z_0 = 2\alpha - 1,$$

and so we must have $\alpha = \frac{1}{2}$. We have

$$\frac{1}{2}y_1 - \frac{1}{2}z_1 = \frac{1}{2}$$
$$\frac{1}{2}y_2 - \frac{1}{2}z_2 = -\frac{1}{2},$$

clearing the denominators gives

$$y_1 - z_1 = 1$$
$$y_2 - z_2 = -1.$$

The only way to satisfy these conditions and have $x_i, y_i \in [0,1]$ is to tatke $y_1 = 1, z_1 = 0$ and $y_2 = 0, z_2 = 1$. We require

$$1 + 0 = \sum_{i=3}^{\infty} 2^{-(i-2)} y_i.$$

For any sequence $w_i \in [0,1]$, we have $\sum_{i=3}^{\infty} 2^{-(i-2)} w_i \in [0,1]$, because $[0,1]$ is complete and convex. Since $y_i \to 0$, there is some k such that $y_i < 1$. We can define $y'_i = y_i$ for $i \neq k$ and $y'_k = 1$. Then

$$\sum_{i=3}^{\infty} 2^{-(i-2)} y_i < \sum_{i=3}^{\infty} 2^{-(i-2)} y'_i \leq 1,$$

so we cannot obtain a $y_i \to 0$ where $\sum_{i=3}^{\infty} 2^{-(i-2)} y_i = 1 = y_1 + y_2$. Therefore $x_i \notin \operatorname{absco}(K)$.

We have $0 \cdot x = 0 \in \operatorname{absco}(K)$ because every absolutely convex set contains 0. We will now show that $\beta x \in \operatorname{absco}(K)$ for every $\beta \in (0,1)$. We have $\beta x = (0, \frac{1}{2}\beta, -\frac{1}{2}\beta, 0, \ldots)$. Take the largest $N \in \mathbb{N}$ (possibly 0) such that $1 - 2^{-N} \leq \beta$. We therefore have

$$1 - 2^{-N} \leq \beta \leq 1 - 2^{-(N+1)},$$

so

$$0 = (1-2^{-N}) - (1-2^{-N}) \leq \alpha - (1-2^{-N}) \leq (1-2^{-(N+1)}) - (1-2^{-N}) = 2^{-(N+1)}.$$

We therefore have $0 \leq 2^{N+1}(\alpha - (1-2^{-N})) \leq 1$. We define

$$y_i = \begin{cases} 1 & \text{if } 3 \leq i \leq N+2 \\ 2^{N+1}(\alpha - (1-2^{-N})) & \text{if } i = N+3 \\ 0 & \text{if } i \geq N+4. \end{cases}$$

By the preceding arguments, $y_i \in [0,1]$ and as it is eventually 0, $y_i \to 0$. We define $y_0 = 1$, $y_1 = \alpha$ and $y_2 = 0$. We then have

$$\begin{aligned} \sum_{i=3} \infty 2^{-(i-2)} y_i &= \sum_{i=3}^{N+2} 2^{-(i-2)} + 2^{-(N+3-2)} 2^{N+1}(\alpha - (1-2^{-N})) + 0 \\ &= 1 - 2^{-N} + \alpha - (1 - 2^{-N}) \\ &= \alpha \\ &= y_1 + y_2, \end{aligned}$$

so $(y_i) \in K$. We can define $z_0 = 1$, $z_1 = 0$, $z_2 = \alpha$ and $z_i = y_i$ for $i \geq 3$. Then similarly to the argument above, we have $(z_i) \in K$. Taking $\alpha = \frac{1}{2}$, we have that for $i = 0$:

$$\frac{1}{2} y_0 - \frac{1}{2} z_0 = \frac{1}{2} - \frac{1}{2} = 0 = \alpha x_0.$$

For $i = 1$:

$$\frac{1}{2}y_1 - \frac{1}{2}z_1 = \frac{1}{2}\alpha = \alpha x_1,$$

and for $i = 2$:

$$\frac{1}{2}y_2 - \frac{1}{2}z_2 = -\frac{1}{2}\alpha = \alpha x_2.$$

For $i \geq 3$, we have $z_i = y_i$ so

$$\frac{1}{2}y_i - \frac{1}{2}z_i = 0 = \alpha x_i,$$

so we have shown that $\alpha(x_i) \in \text{absco}(K)$.

By absolute convexity, we also have $-\alpha x \in \text{absco}(K)$, so for all $\alpha \in (-1, 1)$, we have $\alpha x \in \text{absco}(K)$, and for all other α we have $\alpha x \notin \text{absco}(K)$, because otherwise we could use the convexity of $\text{absco}(K)$ to prove $x \in \text{absco}(K)$, which is false. We therefore have that the intersection of the ray generated by x with $\text{absco}(K)$ is homeomorphic to $(-1, 1)$ and therefore not compact, so $\text{absco}(K)$ is not radially compact. $\square$

Appendix B

Summary

In general, we study pairs of categories, where one consists of state spaces and state transformers, the other of algebras of predicates and predicate transformers, and there is a contravariant equivalence of categories between the two.

Another common thread is the use of probability monads, a categorical way of representing probabilistic maps (by using the Kleisli category) and convex sets (by using the Eilenberg-Moore category). The monads $\mathcal{D}$ and $\mathcal{D}_\infty$ are known as distribution monads. We can describe $\mathcal{D}$ as mapping a set to the set of discrete probability distributions of finite support on it, and $\mathcal{D}_\infty$ as mapping a set to the set of discrete probability distributions of countable support on it. We also consider $\mathcal{E}$, the expectation monad, one version of which assigns a set X to the finitely-additive measures on $\mathcal{P}(X)$. On compact Hausdorff spaces, we use the Radon monad $\mathcal{R}$, which assigns to a compact Hausdorff space X the set of Radon Borel probability measures on it, or equivalently the state space of the C*-algebra $C(X)$.

In the first chapter, we describe a probabilistic version of Gelfand duality, where in the category of commutative C*-algebras we replace *-homomorphisms with maps that are only required to preserve positive elements and the unit, and we replace the category of compact Hausdorff spaces with the Kleisli category of the Radon monad. Kleisli categories of probability monads are a standard way of producing a category of probabilistic mappings.

If we consider non-commutative C*-algebras, a natural category to embed them in when considering only the order structure is order-unit spaces. If we consider the dual spaces of C*-algebras, or alternatively the preduals of W*-algebras, a natural category to embed them in when considering only the order structure is base-norm spaces. The definition of an order-unit space is stable, *i.e.* apparently different definitions used by various authors are equivalent. However, this is not the case for base-norm spaces.

We give the three definitions of base-norm space and examples distinguishing them. We then show that each bounded convex set defines a pre-base-norm space, which is a Banach base-norm space iff the original convex set is sequentially complete. Later, we show that bases of Banach base-norm spaces, equivalently sequentially complete convex sets, are a reflective subcategory of both the categories of Eilenberg-Moore algebras $\mathcal{EM}(\mathcal{D})$ and $\mathcal{EM}(\mathcal{D}_\infty)$.

We show that taking the dual space defines a dual adjunction between pre-base-norm spaces and order-unit spaces, and that this adjunction restricts to an equivalence between reflexive spaces, such as finite-dimensional spaces.

To improve on this duality, we first describe the duality between Banach spaces and Smith spaces. Smith spaces are a kind of space originated by Akbarov that characterize the "bounded weak-*" topology on the dual of a Banach space. We can then define Smith base-norm spaces, which are dual to Banach order-unit spaces, and Smith order-unit spaces, which are dual to Banach base-norm spaces. We can also combine these dualities with the previous adjunction to show that the double dual space is an "enveloping" Smith space, analogous to the enveloping W*-algebra of a C*-algebra.

We go over Świrszcz's theorem that $\mathcal{EM}(\mathcal{R})$ and $\mathcal{EM}(\mathcal{E})$ are equivalent to the category of compact convex subsets of locally convex topological vector spaces. This gives us a characterization of the bases of Smith base-norm spaces and the unit intervals of Smith order-unit spaces without having to consider an embedding in a vector space.

The final chapter is about Gelfand duality for commutative W*-algebras. This involves measure spaces, which are triples (X, Σ, μ), X a set, Σ a σ-algebra on X, and μ a countably additive measure on Σ. For each measure space we can produce a commutative C*-algebra $L^\infty(X, \Sigma, \mu)$ of equivalence classes of measurable functions modulo sets of measure zero. We describe a category of measure spaces such that L^∞ defines a contravariant equivalence of categories with the category of commutative W*-algebras and normal *-homomorphisms. This takes some care, as not every measure space defines a W*-algebra in this way, and not every measurable map defines a normal *-homomorphism.

Appendix C

Samenvatting

In het algemeen, bestuderen we paren van categorieën, waarvan de ene uit toestandruimtes en transformaties van toestanden bestaat, en de andere uit algebra's van predicaten en transformaties van predicaten bestaat, met een equivalentie van categorieën ertussen.

Een andere algemene rode draad is het gebruik van kansmonaden. Die geven een categorische manier om stochastieke afbeeldingen (met behulp van de Kleisli categorie) en convexe verzamelingen (met behulp van de Eilenberg-Moore categorie) te representeren. De monaden $\mathcal{D}$ en $\mathcal{D}_\infty$ heten kansverdelingmonaden. De monade $\mathcal{D}$ beeldt een verzameling af op de verzameling van discrete kansverdelingen met eindige drager, en de monade $\mathcal{D}_\infty$ beeldt een verzameling af op de verzameling van discrete kansverdelingen met aftelbare drager. We beschouwen ook $\mathcal{E}$, de verwachtingmonade, waarvan één uitvoering een verzameling X afbeeldt op de verzameling van eindig additieve kansmaten op $\mathcal{P}(X)$. Op compacte Hausdorff ruimtes, gebruiken we de Radonmonade, die een compacte Hausdorff ruimte X afbeeldt op de ruimte van Radon Borel kansmaten op X, of equivalent, op de toestandruimte van de C*-algebra $C(X)$.

In het eerste hoofdstuk, beschrijven we een stochastische versie van Gelfand-dualiteit, waarbij we geen *-homomorfismen in de categorie van C*-algebra's gebruiken maar lineaire afbeeldingen die positieve elementen en de eenheid bewaren, en waarbij we niet de categorie van compacte Hausdorff ruimtes gebruiken, maar de Kleisli categorie van deze Radonmonad. Kleisli categorieën zijn een standaard manier om een categorie van stochastische afbeedlingen te maken.

Voor het bestuderen van de structuur van de ordening van niet-commutatieve C*-algebras gebruiken we een natuurlijke inbedding in de categorie van ordeëenheidruimten. Op vergelijkbare wijze bestuderen we de structuur van de ordening van de dualen van C*-algebra's, of de predualen van W*-algebra's, via een natuurlijke inbedding in de categorie van basisnormruimten. De definitie van een ordeëenheidruimte is stabiel, *d.w.z.* ogenschijnlijk ongelijke definities die verschillende auteurs gebruiken zijn equivalent. Dit geldt niet voor basisnormruimten.

We geven de drie verschillende definities van het begrip basisnormruimte en we geven voorbeelden die hen onderscheiden. Dan bewijzen we dat elke begrensde convexe verzameling een prebasisnormruimte definiëert, die een Banachbasisnormruimte is dan en slechts dan als de oorspronkelijke convexe verzameling sequentiël volledig is. Tenslotte bewijzen we dat basissen van Banachbasisnormruimtes, of sequentiël volledige convexe verzamelingen, een reflectieve deelcategorie van de categorieën van Eilenberg-Moore algebras $\mathcal{EM}(\mathcal{D})$ en $\mathcal{EM}(\mathcal{D}_\infty)$ vormen.

We tonen aan dat afbeelden naar de duale ruimte een contravariante adjunctie tussen prebasisnormruimtes en ordeëenheidruimtes definiëren, en dat deze adjunctie zich beperkt tot een equivalentie tussen reflexieve ruimtes, bijvoorbeeld eindigdimensionale ruimtes.

Om deze dualiteit te verbeteren, beschrijven we eerst de dualiteit tussen Banachruimtes en Smithruimten. Smithruimten vormen een soort ruimte die oorspronkelijk gedefineerd zijn door Akbarov en die de "begrensde zwakke*" topologie op de duale ruimte van een Banachruimte karakteriseren. Daarmee kunnen we Smithbasisnormruimtes definiëren, die de duale van Banachordeëenheidruimtes zijn, en Smithordeëenheidruimtes, die de duale van Banachbasisnormruimtes zijn. We kunnen ook deze dualiteiten met de vorige adjunctie samenvoegen om te laten zien dat de dubbel duale ruimte een "omhullende" Smithruimte is, zoals de dubbel duale ruimte van een C*-algebra de omhullende W*-algebra is.

We geven een samenvatting van de stelling van Świrszcz, die zegt dat $\mathcal{EM}(\mathcal{R})$ en $\mathcal{EM}(\mathcal{E})$ equivalent zijn aan de categorie van compacte convexe deelverzamelingen van locaal convexe topologische vectorruimtes. Deze stelling geeft ons een karakterisering van de basissen van Smithbasisnormruimtes en van de eenheidintervalen van Smithordeëenheidruimtes, zonder gebruik van een inbedding in een vectorruimte.

Het laatste hoofdstuk gaat over Gelfanddualiteit voor commutatieve W*-algebras. Het gebruikt maatruimten, die drietallen (X, Σ, μ) zijn, waarin X een verzameling is, Σ een σ-algebra op X, en μ een aftelbaar additieve maat op Σ. Voor elke maatruimte, kunnen we een commutatieve C*-algebra $L^\infty(X, \Sigma, \mu)$ van equivalentieklassen van meetbare functies modulo nulverzamelingen produceren. We beschrijven een speciale categorie van maatruimtes zodat L^∞ een contravariante equivalentie van categorieën wordt, met de categorie van commutatieve W*-algebra's en normale *-homomorfismen. Dit vereist zorgvuldigheid, omdat niet alle maatruimten op deze manier een W*-algebra definiëren, en niet alle meetbare afbeeldingen tussen maatruimten normale *-homomorfismen definiëren.

Appendix D

Index of Categories

Name	Objects	Morphisms	Page
AEMod	archimedean effect modules	effect module homomorphisms	45
$\mathbf{AW^*Alg}$	AW*-algebras	normal *-homomorphisms	207
BA	Boolean algebras	Boolean algebra homomorphisms	188
Ban	Banach spaces	bounded linear maps	66
$\mathbf{Ban}_1$	Banach spaces	linear maps of norm ≤ 1	66
BBNS	Banach base-norm spaces	positive trace-preserving maps	61
$\mathbf{BBNS}_{\leq 1}$	Banach base-norm spaces	positive trace-reducing maps	62
BCM	barred commutative monoids	monoid homomorphisms preserving the bar	44
BConv	bounded convex subsets of locally convex topological vector spaces	affine maps	67
BEMod	Banach effect modules	effect module homomorphisms	45
BNS	base-norm spaces	positive trace-preserving maps	61
$\mathbf{BNS}_{\leq 1}$	base-norm spaces	positive trace-reducing maps	62
BOUS	Banach order-unit spaces	positive unital maps	45

$\mathbf{BOUS}_{\leq 1}$	Banach order-unit spaces	positive subunital maps	45
Cat	large categories	functors	30
$\mathbf{CAW^*Alg}$	commutative AW*-algebras	normal *-homomorphisms	207
CBA	complete Boolean algebras	Boolean algebra homomorphisms preserving all joins and meets	182
CBConv	sequentially complete bounded convex subsets of locally convex topological vector spaces	affine maps	95
CBConvBan	closed bounded convex subsets of Banach spaces	affine maps	95
CCL	compact convex subsets of locally convex topological vector spaces	continuous affine maps	42
$\mathbf{CC^*Alg}$	commutative C*-algebras	*-homomorphisms	38
$\mathbf{CC^*Alg}_{\mathrm{P}\leq 1}$	commutative C*-algebras	positive subunital maps	38
$\mathbf{CC^*Alg}_{\mathrm{PU}}$	commutative C*-algebras	positive unital maps	38
CEMod	effect modules that are compact convex subsets of locally convex topological vector spaces	continuous effect module homomorphisms	141
$\mathbf{CEMod}_{\mathcal{E}}$	effect modules that are $\mathcal{E}$-algebras	effect module homomorphisms that are also $\mathcal{E}$-algebra homomorphisms	176
$\mathbf{CEMod}_{\mathcal{R}}$	effect modules with a compact Hausdorff topology that are $\mathcal{R}$-algebras	effect module homomorphisms that are also $\mathcal{R}$-algebra homomorphisms	176
CHaus	compact Hausdorff spaces	continuous maps	33
CSL	compact meet semilattices	continuous lattice homomorphisms	177

$\mathbf{C^*Alg}$	C*-algebras	*-homomorphisms	38
$\mathbf{C^*Alg}_{\mathrm{CPU}}$	C*-algebras	completely positive unital maps	159
$\mathbf{C^*Alg}_{\mathrm{P}\leq 1}$	C*-algebras	positive subunital maps	38
$\mathbf{C^*Alg}_{\mathrm{PU}}$	C*-algebras	positive unital maps	38
$\mathbf{CW^*Alg}$	commutative W*-algebras	normal *-homomorphisms	159
$\mathbf{CW^*Alg}_{\mathrm{PU}}$	commutative W*-algebras	normal positive unital maps	159
EA	effect algebras	effect algebra homomorphisms	43
$\mathcal{EM}(T)$	T-algebras	T-algebra morphisms	28
EMod	effect modules over $[0, 1]$	effect module homomorphisms	43
$\mathbf{FdCC^*Alg}_{MIU}$	finite dimensional commutative C*-algebras	*-homomorphisms	39
$\mathbf{FdCC^*Alg}_{PU}$	finite dimensional commutative C*-algebras	positive unital maps	39
$\mathbf{FdC^*Alg}_{MIU}$	finite dimensional C*-algebras	*-homomorphisms	39
$\mathbf{FdC^*Alg}_{PU}$	finite dimensional C*-algebras	positive unital maps	39
FinSet	finite sets	functions	49
HypStonean	hyperstonean spaces	continuous open maps	213
$\mathcal{K}\ell(T)$	objects of underlying category	morphisms $X \to T(X)$	29
$\mathcal{K}\ell_{\mathbb{N}}(T)$	finite cardinals	morphisms $n \to T(m)$	29
$\mathcal{M}eas$	compact strictly localizable complete measure spaces (X, Σ, μ) such that if $\mu(X) = 0$, $X = \emptyset$	normal measurable maps	231
Meas	See $\mathcal{M}eas$	equivalence classes of normal measurable maps, equivalently according to $\mathbf{Meas}_{\mathrm{BA}}$ or $\mathbf{Meas}_{L^\infty}$	254
$\mathbf{Meas}_{\mathrm{BA}}$	See $\mathcal{M}eas$	equivalence classes of normal measurable maps under $f \sim g \Leftrightarrow \mathrm{BA}(f) = \mathrm{BA}(g)$	254

$\mathbf{Meas}_{L^\infty}$	See $\mathcal{M}eas$	equivalence classes of normal measurable maps under $f \sim g \Leftrightarrow L^\infty(f) = L^\infty(g)$	254
MeasAlg	measure algebras	complete Boolean homomorphisms	216
$\mathcal{M}és$	measurable spaces (X, Σ)	measurable functions	217
$\mathbf{Monad}(\mathcal{C})$	monads on the category $\mathcal{C}$	monad morphisms	30
Normed	normed spaces	bounded linear maps	66
$\mathbf{Normed}_1$	normed spaces	linear maps of norm ≤ 1	66
OUS	order-unit spaces	positive unital maps	45
$\mathbf{OUS}_{\leq 1}$	order-unit spaces	positive subunital maps	45
Poset	partially ordered sets	monotone maps	189
poVectu	partially ordered vector spaces with strong order unit	positive unital maps	44
PreBNS	pre-base-norm spaces	positive trace-preserving maps	61
$\mathbf{PreBNS}_{\leq 1}$	pre-base-norm spaces	positive trace-reducing maps	62
Predual	Preduals of W*-algebras	positive trace-preserving maps	75
RBNS	reflexive base-norm spaces	positive trace-preserving maps	106
ROUS	reflexive order-unit spaces	positive unital maps	106
SBNS	Smith base-norm spaces	continuous positive trace-preserving maps	131
$\mathbf{SBNS}_{\leq 1}$	Smith base-norm spaces	continuous positive trace-reducing maps	131
Set	sets	functions	29
Smith	Smith spaces	continuous linear maps	111
$\mathbf{Smith}_1$	Smith spaces	continuous linear maps of norm ≤ 1	111
SOUS	Smith order-unit spaces	continuous positive unital maps	140
$\mathbf{SOUS}_{\leq 1}$	Smith order-unit spaces	continuous positive subunital maps	140
Stone	Stone spaces	continuous maps	188

Stonean	stonean spaces	continuous open maps	213
Top	topological spaces	continuous maps	33
Vect	vector spaces over $\mathbb{C}$	$\mathbb{C}$-linear maps	37
$\mathbf{W^*Alg}$	W*-algebras	normal *-homomorphisms	159
$\mathbf{W^*Alg}_{\mathrm{P}\leq 1}$	W*-algebras	normal positive subunital maps	159
$\mathbf{W^*Alg}_{\mathrm{PU}}$	W*-algebras	normal positive unital maps	159

Index

Bibliography

[1] Samson Abramsky and Bob Coecke. A Categorical Semantics of Quantum Protocols. In K. Engesser, Dov M. Gabbai, and D. Lehmann, editors, *Handbook of Quantum Logic and Quantum Structures*, pages 261–323. North Holland, Elsevier, Computer Science Press, 2009. 1.1

[2] Sergei S. Akbarov. Pontryagin Duality in the Theory of Topological Vector Spaces and in Topological Algebra. *Journal of Mathematical Sciences*, 113(2):179–349, 2003. 3.1, 3.1, 3.2, 3.2

[3] Sergei S. Akbarov. Holomorphic Functions of Exponential Type and Duality for Stein Groups with Algebraic Connected Component of Identity. *Journal of Mathematical Sciences*, 162(4):459–586, 2009. 3.1

[4] Erik M. Alfsen. *Compact Convex Sets and Boundary Integrals*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1971. 2.2.1, 2.2.3, 2.2.3, 2.5.1, 3.3.1, 4.2

[5] Erik M. Alfsen, Harald Hanche-Olsen, and Frederic W. Shultz. State Spaces of C^*-algebras. *Acta Mathematica*, 144(1):267–305, 1980. 3.6

[6] Erik M. Alfsen and Frederic W. Shultz. *State Spaces of Operator Algebras*. Birkhäuser, 2001. 1.2.1, 2.2.1, 2.2.3, 2.2.3, 2.3, 2.4.2, 2.5.1, 5.2.4, A.5

[7] Erik M. Alfsen and Frederic W. Shultz. *Geometry of State Spaces of Operator Algebras*. Birkhäuser, 2003. 2.5.1

[8] Leonard Asimow and Alan J. Ellis. *Convexity Theory and its Applications in Functional Analysis*. Number 16 in L.M.S. Monographs. Academic Press, 1980. 2.2.3, 3.2, 3.2

[9] Howard Barnum and Alexander Wilce. Ordered Linear Spaces and Categories as Frameworks for Information-Processing Characterizations of Quantum and Classical Theory. `http://arxiv.org/abs/0908.2354`, 2009. 2.1

[10] Michael Barr and Charles Wells. Toposes, Triples and Theories. *Reprints in Theory and Applications of Categories*, 12:1–288, 2005. 0.4.1

[11] Jonathan Barrett. Information Processing in Generalized Probabilistic Theories. `http://arxiv.org/abs/quant-ph/0508211`, 2006. 2.1

[12] Jonathan Barrett. Information Processing in Generalized Probabilistic Theories. *Physical Review A*, 75:032304, Mar 2007. 2.1

[13] Sterling K. Berberian. *Baer *-Rings*, volume 195 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1972. 5.2.5, 5.2.5

[14] Guram Bezhanishvili. Stone duality and Gleason covers through de Vries duality. *Topology and its Applications*, 157(6):1064 – 1080, 2010. 5.1, 5.3

[15] Patrick Billingsley. *Convergence of Probability Measures*. John Wiley and Sons, 1968. 1.5, 5.4.1

[16] Garrett Birkhoff. *Lattice Theory*, volume XXV of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1948. iii

[17] Francis Borceux. *Handbook of Categorical Algebra*, volume 50, 51 and 52 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1994. 0.4, 0.4.1, 0.4.1

[18] Nicolas Bourbaki. *Topological Vector Spaces*. Springer, 1987. 0.3, 0.3, 0.3.1, 0.3.1, 3.2, 2

[19] Nicolas Bourbaki. *General Topology*. Ettore Majorana International Science. Springer, 1998. 2, 3.2, 2, 3.2, 3.5, 5.2.5

[20] Kenta Cho. Semantics for a Quantum Programming Language by Operator Algebras. *New Generation Computing*, 34(1):25–68, 2016. (document), 5.9

[21] Johann Cigler, Viktor Losert, and Peter Michor. *Banach Modules and Functors on Categories of Banach Spaces*, volume 46 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., 1979. 3.1

[22] Alain Connes. Non-commutative Differential Geometry. *Publications Mathématiques de l'IHES*, 62:41–144, 1985. 5.1

[23] Alain Connes. *Noncommutative Geometry*. Academic Press, San Diego, CA, 1994. 5.1

[24] John B. Conway. *A Course In Functional Analysis, Second Edition*, volume 96 of *Graduate Texts in Mathematics*. Springer Verlag, 1990. 0.1, 0.1, 0.1, 0.1, 0.3.1, 1.5, 2.4.1, 2.5.1, 2.5.1, 3.2, 4.2, 4.2, 5.2.4, A.6

[25] John Dauns. Categorical W^*-Tensor Product. *Transactions of the American Mathematical Society*, 166:pp. 439–456, 1972. 3.5

[26] E. Brian Davies and John T. Lewis. An Operational Approach to Quantum Probability. *Comm. Math. Phys.*, 17(3):239–260, 1970. 2.1

[27] Ellie d'Hondt and Prakash Panangaden. Quantum Weakest Preconditions. *Mathematical Structures in Computer Science*, 16(3):429–451, 2006. (document)

[28] Jacques Dixmier. Sur un théorème de Banach. *Duke Mathematical Journal*, 15(4):1057–1071, 12 1948. 3.2

[29] Jacques Dixmier. *C*-algebras*, volume 15 of *North-Holland Mathematical Library*. North-Holland Publishing Company, 1977. 1.2, ii, 1.2.1, 1.3, 1.5, 2.3, 3.6, ii

[30] Jacques Dixmier. *Von Neumann Algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Company, 1981. 5.2.5

[31] Richard M. Dudley. *Real Analysis and Probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2004. 5.4.3

[32] Nelson Dunford and Jacob T. Schwartz. *Linear Operators, Part 1: General Theory*. Number VII in Pure and Applied Mathematics. Interscience Publishers, 1957. 3.2, 3.2, 3.2, 3.2, 3.2, A.6

[33] Anatolij Dvurečenskij and Sylvia Pulmannová. *New Trends in Quantum Structures*. Kluwer Acad. Publ., Dordrecht, 2000. 1.2.1

[34] Christopher M. Edwards. The Operational Approach to Algebraic Quantum Theory I. *Communications in Mathematical Physics*, 16(3):207–230, 1970. 1.2.1, 2.1, 2.3, 2.4

[35] Alan J. Ellis. The Duality of Partially Ordered Normed Linear Spaces. *Journal of the London Mathematical Society*, s1-39(1):730–744, 1964. 2.5.1, 3.1

[36] Alan J. Ellis. Minimal Decompositions in Base Normed Spaces. In Alois Hartkämper and Hans Neumann, editors, *Foundations of Quantum Mechanics and Ordered Linear Spaces: Advanced Study Institute Marburg 1973*, pages 30–32. Springer Berlin Heidelberg, 1974. 2.2.3, A.6

[37] Vitaly V. Fedorchuk. Probability Measures in Topology. *Russian Mathematical Surveys*, 46(1):45–93, 1991. 1.5, 4.2

[38] David J. Foulis and Mary K. Bennett. Effect Algebras and Unsharp Quantum Logics. *Found. Physics*, 24(10):1331–1352, 1994. 1.2.1

[39] David H. Fremlin. Measure Theory, Volume 1. `http://www.essex.ac.uk/maths/people/fremlin/mt.htm`, 2000. 5.4.1, 5.4.1

[40] David H. Fremlin. Measure Theory, Volume 2. http://www.essex.ac.uk/maths/people/fremlin/mt.htm, 2001. 5.4.1, 5.4.1, i, ii, iii, 5.4.1, 5.4.1, 5.4.1, 5.4.3, 5.4.3, 5.4.3, 5.4.3, 5.5, 5.5, 5.7.1

[41] David H. Fremlin. Measure Theory, Volume 3. http://www.essex.ac.uk/maths/people/fremlin/mt.htm, 2002. 5.1, 5.3, 5.3, 5.4.1, 5.4.1, 5.4.1, 5.6, 5.6, 5.6, 5.6, 5.6, 5.7.1, 5.7.1, 5.7.2, 5.8, 5.10

[42] David H. Fremlin. Measure Theory, Volume 4. http://www.essex.ac.uk/maths/people/fremlin/mt.htm, 2003. 1.5, 5.4.1, 5.4.1, 5.10

[43] Tobias Fritz. A Presentation of the Category of Stochastic Matrices. http://arxiv.org/abs/0902.2554, 2009. 1.4

[44] Robert Furber and Bart Jacobs. From Kleisli Categories to Commutative C*-algebras: Probabilistic Gelfand Duality. In Reiko Heckel and Stefan Milius, editors, *Algebra and Coalgebra in Computer Science*, volume 8089 of *Lecture Notes in Computer Science*, pages 141–157. Springer Berlin Heidelberg, 2013. 1

[45] Robert Furber and Bart Jacobs. From Kleisli Categories to Commutative C*-algebras: Probabilistic Gelfand Duality. *Logical Methods in Computer Science*, 11(2), 2014. 1, 3.6, 4, 4.2

[46] Robert Furber, Mathys Rennela, and Sam Staton. Infinite-Dimensionality in Quantum Foundations: W*-algebras as Presheaves over Matrix Algebras. http://www.cs.ru.nl/M.Rennela/papers/qpl16.pdf, 2016. 1.2

[47] Michèle Giry. A Categorical Approach to Probability Theory. In B. Banaschewski, editor, *Categorical Aspects of Topology and Analysis*, pages 68–85. Springer, 1980. 1.1, 5.4.1

[48] Stanley Gudder and Sylvia Pulmannovà. Representation Theorem for Convex Effect Algebras. *Commentationes Mathematicae Universitatis Carolinae*, 39(4):645 – 659, 1998. 1.2.1, 3.1

[49] Paul R. Halmos. *Measure Theory*, volume 18 of *Graduate Texts in Mathematics*. Springer, 1950. 1.5, i, 1

[50] Paul R. Halmos. *Lectures on Boolean Algebras*. Van Nostrand Mathematical Studies. D. van Nostrand Company, Inc., 1963. 5.2.5

[51] Teiko Heinosaari and Mario Ziman. *The Mathematical Language of Quantum Theory. From Uncertainty to Entanglement*. Cambridge Univ. Press, 2012. 1.2.1

[52] Chris Heunen. Reply to *Reference for the Gelfand-Neumark Theorem for Commutative von Neumann Algebras*. http://mathoverflow.net/a/78322, 2011. 5.1, 5.3, 5.3

[53] Chris Heunen and Bert Lindenhovius. Domains of Commutative C*-algebras. In *Proceedings of the 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 450–461, 2015. 5.10

[54] Chris Heunen and Manuel L. Reyes. Active Lattices Determine AW*-algebras. *Journal of Mathematical Analysis and Applications*, 416(1):289 – 313, 2014. 5.3, 5.3

[55] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Separability of Mixed States: Necessary and Sufficient Conditions. *Physics Letters A*, 223(12):1 – 8, 1996. 1.2

[56] Bart Jacobs. Convexity, Duality and Effects. In Cristian S. Calude and Vladimiro Sassone, editors, *Theoretical Computer Science*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 1–19. Springer Berlin Heidelberg, 2010. 0.4.1, 1.2.1, 2.1, 2.5, 3.6

[57] Bart Jacobs. Involutive Categories and Monoids, with a GNS-correspondence. *Found. of Physics*, 42(7):874–895, 2012. 1.2

[58] Bart Jacobs. Measurable Spaces and their Effect Logic. In *Logic in Computer Science*. IEEE, Computer Science Press, 2013. 4.2

[59] Bart Jacobs. Lectures on Kadison Duality. `http://www.cs.ru.nl/B.Jacobs/TALKS/kadison-lectures-6up.pdf`, 2014. 3.1, 3.3.1

[60] Bart Jacobs. A Recipe for State-and-Effect Triangles. In Lawrence S. Moss and Pawel Sobocinski, editors, *6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015)*, volume 35 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 116–129, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. 3.6

[61] Bart Jacobs. New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic. *Logical Methods in Computer Science*, 11(3), 2015. 3.6

[62] Bart Jacobs and Jorik Mandemaker. Coreflections in Algebraic Quantum Logic. *Found. of Physics*, 42(7):932–958, 2012. 1.2.1

[63] Bart Jacobs and Jorik Mandemaker. The Expectation Monad in Quantum Foundations. In Bart Jacobs, Peter Selinger, and Bas Spitters, editors, *Quantum Physics and Logic (QPL) 2011*, volume 95 of *Elect. Proc. in Theor. Comp. Sci.*, pages 143–182, 2012. 1.2.1, 1.2.1, 1.2.1, 1.2.1, 1.4, 1.4, 1.4.1, 1.4, 1.5, 2.1, 3.1, 4, 4.3

[64] Bart Jacobs, Jorik Mandemaker, and Robert Furber. The Expectation Monad in Quantum Foundations. *Information and Computation*, pages –, 2016. 0, 1, 4, 4.2

[65] Bart Jacobs and Bram Westerbaan. An Effect-Theoretic Account of Lebesgue Integration. In *The Thirty-First Conference on the Mathematical Foundations of Programming Semantics (MFPS XXXI)*, 2015. 5.2.4

[66] Graham Jameson. *Ordered Linear Spaces*, volume 141 of *Lecture Notes in Mathematics*. Springer, 1970. 1.2.1

[67] Thomas J. Jech. *Set Theory*. Springer, 3rd millennium edition edition, 2003. 5.5

[68] Peter Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982. 0.4.1, 1.2, 5.2.1, iii

[69] Richard Kadison and John Ringrose. *Fundamentals of the Theory of Operator Algebras*. Academic Press, 1983. 1.2, ii, 3.6

[70] Richard V. Kadison. *A Representation Theory for Commutative Topological Algebra*, volume 7 of *Memoirs of the AMS*. AMS, 1951. 3.1, 3.3.1

[71] Richard V. Kadison. Transformations of States in Operator Theory and Dynamics. *Topology*, 3, Supplement 2(0):177 − 198, 1965. 3.3.1

[72] Nigel J. Kalton and N. Tenney Peck. A Re-examination of the Roberts Example of a Compact Convex Set without Extreme Points. *Mathematische Annalen*, 253(2):89–101, 1980. 4.2

[73] Alexander S. Kechris. *Classical Descriptive Set Theory*. Number 156 in Graduate Texts in Mathematics. Springer, 1995. 5.4.1

[74] Klaus Keimel. Abstract Ordered Compact Convex Sets and Algebras of the (sub)Probabilistic Powerdomain Monad over Ordered Compact Spaces. *Algebra and Logic*, 48(5):330–343, 2009. 4.2

[75] John L. Kelley. *General Topology*. Number 27 in Graduate Texts in Mathematics. Springer, 1955. 3.2, 5.4.1, A.2

[76] Heinz König. Theory and Applications of Superconvex Spaces. In Rainer Nagel, Ulf Schlotterbeck, and Manfred P. H. Wolff, editors, *Aspects of Positivity in Functional Analysis*, volume 122 of *North-Holland Mathematics Studies*, pages 79–118. North-Holland, 1986. 2.4

[77] Heinz König and Gerd Wittstock. Superconvex Sets and σ-convex Sets, and the Embedding of Convex and Superconvex Spaces. *Note di Matematica*, 10(Suppl. 2):343–362, 1990. 2.4

[78] Bernard O. Koopman. Hamiltonian Systems and Transformation in Hilbert Space. *Proceedings of the National Academy of Sciences*, 17(5):315–318, 1931. 5.1

[79] Andre Kornell. Quantum Collections. `http://arxiv.org/abs/1202.2994`, 2012. 5.10

[80] Dexter Kozen. A Probabilistic PDL. *Journal of Computer and System Sciences*, 30(2):162 – 178, 1985. 1.1

[81] Joachim Lambek and Philip Scott. *Introduction to Higher Order Categorical Logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1986. 2.5.1

[82] Klaas P. Landsman. *Mathematical Topics Between Classical and Quantum Mechanics*. Springer Monographs in Mathematics. Springer, 1998. 3.6

[83] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer Verlag, 1971. 0.4, 0.4, 0.4, 0.4, 0.4.1, 0.4.1, 3.2, 5.2.4

[84] Tom Leinster. *Higher Operads, Higher Categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004. 3

[85] Jacob A. Lurie. Von Neumann Algebras (261y). `http://www.math.harvard.edu/~lurie/261y.html`. 5.1

[86] Hans Maassen. Quantum Probability and Quantum Information Theory. In Fabio Benatti, Mark Fannes, Roberto Floreanini, and Dimitri Petritis, editors, *Quantum Information, Computation and Cryptography*, number 808 in Lect. Notes Physics, pages 65–108. Springer, Berlin, 2010. 1.4

[87] Ernest G. Manes. A Triple-Theoretic Construction of Compact Algebras. In Beno Eckmann, editor, *Seminar on Triples and Categorical Homology Theory*, number 80 in Lect. Notes Math., pages 91–118. Springer, Berlin, 1969. 0.4.1, 1.2

[88] Ernest G. Manes. *Algebraic Theories*, volume 26 of *Graduate Texts in Mathematics*. Springer, 1974. 0.4.1

[89] Edward Marczewski. On Compact Measures. *Fundamenta Mathematicae*, 40:113–124, 1953. 5.4.1

[90] Michael Mislove. Probabilistic Monads, Domains and Classical Information. In Elham Kashefi, Jean Krivine, and Femke van Raamsdonk, editors, Proceedings 7th International Workshop on *Developments of Computational Methods,* Zurich, Switzerland, 3rd July 2011, volume 88 of *Electronic Proceedings in Theoretical Computer Science*, pages 87–100. Open Publishing Association, 2012. 1.2, 1.5

[91] Michael Mislove, Joel Ouaknine, Dusko Pavlovic, and James Worrell. Duality for Labelled Markov Processes. In Igor Walukiewicz, editor, *Foundations of Software Science and Computation Structures*, volume 2987 of

Lecture Notes in Computer Science, pages 393–407. Springer Berlin Heidelberg, 2004. 1.1

[92] Eugenio Moggi. Notions of Computation and Monads. *Inf. & Comp.*, 93(1):55–92, 1991. 0.4.1

[93] Leopoldo Nachbin. *The Haar Integral*. University Series in Higher Mathematics. D. van Nostrand Company, 1964. 5.4.1

[94] Rainer J. Nagel. Order Unit and Base Norm Spaces. In Alois Hartkämper and Hans Neumann, editors, *Foundations of Quantum Mechanics and Ordered Linear Spaces*, volume 29 of *Lecture Notes in Physics*, pages 23–29. Springer Berlin Heidelberg, 1974. 2.2.3, 2.2.3

[95] Isaac Namioka and Robert R. Phelps. Tensor Products of Compact Convex Sets. *Pacific Journal of Mathematics*, 31(2):469–480, 1969. 2.1

[96] Kung-fu Ng. On a Theorem of Dixmier. *Mathematica Scandinavica*, 29:279–280, 1971. 3.2

[97] nLab. (eso+full, faithful) factorization system. `http://ncatlab.org/nlab/revision/%28eso%2Bfull%2C+faithful%29+factorization+system/10`, 2012. 5.8

[98] Jan K. Pachl. Disintegration and Compact Measures. *Mathematica Scandinavica*, 43:157–168, 1978. 5.4.1

[99] Prakash Panangaden. *Labelled Markov Processes*. World Scientific, 2009. 1

[100] Dmitri Pavlov. Is There a Category Structure one can Place on Measure Spaces so that Category-Theoretic Products Exist? MathOverflow. `http://mathoverflow.net/q/49542` (version: 2011-01-08). 5.3

[101] Joan Wick Pelletier and Jiři Rosický. On the Equational Theory of C^*-algebras. *Algebra Universalis*, 30:275–284, 1993. 1.1

[102] Robert R. Phelps. *Lectures on Choquet's Theorem*, volume 1757 of *Lecture Notes in Mathematics*. Springer, 2001. 4.2

[103] Albrecht Pietsch. *Nuclear Locally Convex Spaces*. Number 66 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, 1972. 0.1

[104] Gordon D. Plotkin. Dijkstra's Predicate Transformers and Smyth's Powerdomains. In Dines Bjørner, editor, *Abstract Software Specifications: 1979 Copenhagen Winter School January 22 – February 2, 1979 Proceedings*, pages 527–553. Springer Berlin Heidelberg, Berlin, Heidelberg, 1980. (document)

[105] Dieter Pumplün. The Universal Compactification of Topological Convex Sets and Modules. *Journal of Convex Analysis*, 8(1):255–268, 2001. 3.5

[106] Dieter Pumplün. The Metric Completion of Convex Sets and Modules. *Results in Mathematics*, 41(3-4):346–360, 2002. 2.4.1, 2.4.1, 3.4.1

[107] Dieter Pumplün and Helmut Röhrl. Banach Spaces and Totally Convex Spaces I. *Communications in Algebra*, 12(8):953–1019, 1984. 0.1, 2.4.1

[108] Dieter Pumplün and Helmut Röhrl. Banach Spaces and Totally Convex Spaces II. *Communications in Algebra*, 13(5):1047–1113, 1985. 0.1

[109] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, 1980. 5.2.4

[110] Mathys Rennela. Operator Algebras in Quantum Computation. Master's thesis, Université Paris 7 Diderot, 2013. `http://arxiv.org/abs/1510.06649`. (document), 3.6

[111] Mathys Rennela and Sam Staton. Complete Positivity and Natural Representation of Quantum Computations. In *Proc. Mathematical Foundations of Programming Semantics XXXI, MFPS 2015*, ENTCS, 2015. 3.6

[112] James W. Roberts. A Compact Convex Set with no Extreme Points. *Studia Mathematica*, 60(3):255–266, 1977. 4.2

[113] Gerd Rodé. Superkonvexität und schwache Kompaktheit. *Archiv der Mathematik*, 36(1):62–72, 1981. 2.4

[114] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill Book Company, 1987. Third, International edition. 1.1, 1.5, 5.4.1, 5.4.2

[115] Bernard Russo and Henry A. Dye. A Note on Unitary Operators in C^*-algebras. *Duke Math. J.*, 33:413–416, 1966. 1.2

[116] Kazuyuki Saitô and J. D. Maitland Wright. On Defining AW*-algebras and Rickart C*-algebras. *The Quarterly Journal of Mathematics*, 2015. i

[117] Shoichiro Sakai. *C^*-algebras and W^*-algebras*, volume 60 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1971. ii, 2.3

[118] Helmut H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer Verlag, 1966. 0.1, 1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3.1, 0.3.1, 0.3.1, 2.2.2, 2.2.3, 3.2.11, 3.2, 3.2, 3.2, 3.2, 3.2, 4.2

[119] Irving E. Segal. Equivalences of Measure Spaces. *American Journal of Mathematics*, 73(2):pp. 275–313, 1951. 5.1, 5.4.1, 5.4.3

[120] Zbigniew Semadeni. *Monads and their Eilenberg-Moore Algebras in Functional Analysis*, volume 33 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University at Kingston, Ontario, Canada, 1973. 4.2, 4.3

[121] Roman Sikorski. *Boolean Algebras.* Number 25 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1969. i, ii

[122] Alex Simpson. Category-theoretic Structure for Conditional Independence. http://homepages.inf.ed.ac.uk/als/Talks/cambridge14.pdf, 2014. 5.1

[123] Marianne Freundlich Smith. The Pontrjagin Duality Theorem in Linear Spaces. *Annals of Mathematics*, 56(2):pp. 248–253, 1952. 3.1

[124] Michael B. Smyth. Power Domains and Predicate Transformers: A Topological View. In Josep Diaz, editor, *Automata, Languages and Programming: 10th Colloquium Barcelona, Spain, July 18–22, 1983*, pages 662–675. Springer Berlin Heidelberg, Berlin, Heidelberg, 1983. (document)

[125] Ross Street. The Formal Theory of Monads. *Journ. of Pure & Appl. Algebra*, 2:149–169, 1972. 0.4.1, 1.5.2

[126] Tadeusz Świrszcz. Monadic Functors and Convexity. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Math. Astr. et Phys.*, 22(1):39–42, 1974. 1.2, 1.5, 1.5, 4.2, 4.2, 4.3

[127] Tadeusz Świrszcz. Monadic Functors and Categories of Convex Sets. Institute of Mathematics of the Polish Academy of Sciences, Preprint 70, 1975. 0.4.1

[128] Masamichi Takesaki. *Theory of Operator Algebra*, volume 1. Springer Verlag, 1979. 2.3, 2.5.1, 3.6, 5.1, 5.2.4, 5.2.5, 5.2.5, 5.2.5

[129] Masamichi Takesaki. *Theory of Operator Algebras II.* Encyclopaedia of Mathematical Sciences. Springer Verlag, 2001. 5.6

[130] Alexandra Ionescu Tulcea and Cassius Ionescu Tulcea. *Topics in the Theory of Lifting*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1969. 5.7.1, 5.10

[131] John von Neumann. Zur Operatorenmethode In Der Klassischen Mechanik. *Annals of Mathematics*, 33(3):pp. 587–642, 1932. 5.1

[132] Bram Westerbaan. Quantum Programs as Kleisli Maps. http://arxiv.org/abs/1501.01020. 1.1

[133] Alexander Wilce. Tensor Products in Generalized Measure Theory. *International Journal of Theoretical Physics*, 31(11):1915–1928, 1992. 2.1

[134] Oswald Wyler. Algebraic Theories of Continuous Lattices. In Bernhard Banaschewski and Rudolf-Eberhard Hoffmann, editors, *Continuous Lattices*, volume 871 of *Lecture Notes in Mathematics*, pages 390–413. Springer Berlin Heidelberg, 1981. 4.5