Scott Continuity in Generalized Probabilistic Theories

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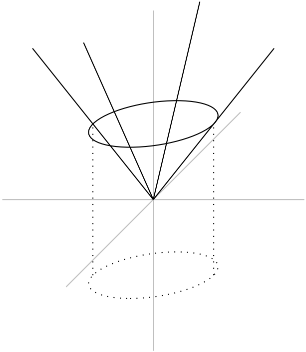
Overview

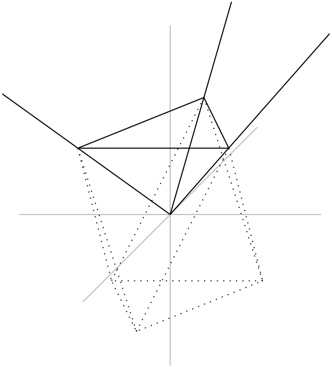
- Background on Generalized Probabilistic Theories
- Background on Scott Continuity and Domain Theory
- Counterexamples

Convex Sets

- What kind of structure does a (mixed) state space X have?
- Mixing states.
- Many axiomatizations:
  - In terms of operations $(x, y) \mapsto x +_\alpha y$, subject to axioms.
  - In terms of an operation $\mathcal{D}(X) \to X$.
  - In terms of convex subsets X of vector spaces E.
  - The base of a base-norm space $(E, E_+, \tau : E \to \mathbb{R})$.

Examples





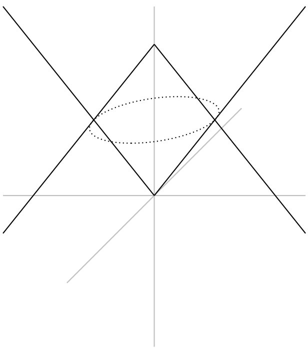
- Density matrices and $\ell^1(3)$.
- Other examples such as the square bit/boxworld.
- Convex sets not embeddable in vector spaces: $\{0, \infty\}$ and $[0, \infty]$.

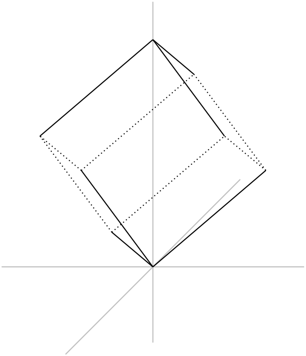
Effects

- Morphisms of state spaces are required to be affine.
- For base-norm spaces, affine morphisms extend to linear ones.
  - Special case: Affine maps $\mathcal{DM}(\mathcal{H}_1) \to \mathcal{DM}(\mathcal{H}_2)$ extend to positive trace-preserving maps.
- Effects on X, $\mathcal{E}(X)$ are affine maps $X \to \mathcal{D}(2) \cong [0, 1]$.
- They live inside a vector space $\mathcal{E}_\pm(X)$, the set of bounded affine functions $X \to \mathbb{R}$.
- $\mathcal{E}_\pm(\mathcal{DM}(\mathcal{H})) \cong \mathrm{SA}(\mathcal{H})$ and $\mathcal{E}_\pm(\mathcal{D}(X)) \cong \ell^\infty(X)$.
- $\mathcal{E}(\mathcal{DM}(\mathcal{H})) = \mathrm{Ef}(\mathcal{H})$ and $\mathcal{E}(\mathcal{D}(X)) = [0, 1]^X$.

Abstract Effects

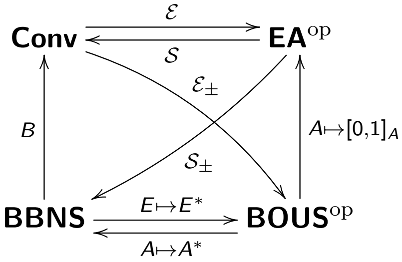
- Many axiomatizations:
  - Test spaces
  - Orthomodular lattices
  - Effect algebras: $(A, \ovee, \text{-}^{\perp})$
  - Convex effect algebras
  - Order-unit spaces: (A, A_+, u).
- Examples of order-unit spaces:





States

- Morphisms of effect algebras preserve the addition, complement and unit.
- $[0,1]$ is an effect algebra, and $\mathcal{S}(A)$ is the set of maps $A \to [0,1]$.
- $\mathcal{S}(A)$ is the base of a base-norm space $\mathcal{S}_{\pm}(A)$.
- For an order-unit space A, $\mathcal{S}_{\pm}([0,1]_A) \cong A^*$.



- The natural map $X \mapsto \mathcal{S}(\mathcal{E}(X))$ is an isomorphism iff $X \cong B(E)$ for E a reflexive base-norm space.
- The natural map $A \mapsto \mathcal{E}(\mathcal{S}(A))$ is an isomorphism iff $A \cong [0,1]_B$ for B a reflexive order-unit space.

State-Effect Duality II

- This does not work for $\mathcal{DM}(L^2(\mathbb{R}^n))$ or any kind of infinite-dimensional quantum mechanics.
- Fix: Break it into two dualities.
- Use the weak-* topology to make $\mathcal{S}(A)$ compact, take continuous effects $\mathcal{CE}$, get a duality $\mathbf{BOUS}^{\mathrm{op}} \simeq \mathbf{SBNS}$.
- Use the weak-* topology to make $\mathcal{E}(X)$ compact, take continuous states $\mathcal{CE}$, get a duality $\mathbf{BBNS} \simeq \mathbf{SOUS}^{\mathrm{op}}$.

Domain Theory

- In mathematics, start with sets, then describe morphisms.
- In computer science, we have morphisms (described syntactically) and we want to find out what the sets are (domains).
- Use dcpos $(D, \leq)$.
  - A directed set $S \subseteq D$ is one in which every pair $x, y \in S$ has an upper bound in S.
  - Directed-complete means each directed set has a *least* upper bound.
- Morphisms: Scott-continuous maps.
- Can interpret recursive functions by iterating to a fixed point: $\bot \leq f(\bot) \leq f(f(\bot)) \cdots$.
- First models of untyped λ-calculus were obtained by finding a dcpo D such that $D \cong [D \to D]$.

Domain Theory and Quantum

- The spaces $B(\mathcal{H})$ and $L^\infty(X, \mu)$, and any von Neumann algebra, are bounded-directed complete.
- So $\mathcal{E}(\mathcal{DM}(\mathcal{H}))$ and $\mathcal{E}(\mathcal{DF}(X, \mu))$ are dcpos, as is $[0, 1]_A$ for any von Neumann algebra A.
- Scott continuous maps $[0, 1]_A \to [0, 1]_B$ form a dcpo.
- $f : [0, 1]_A \to [0, 1]_B$ is (weak-*) continuous iff it is Scott continuous.
- Key fact: a state $\phi : B(\mathcal{H}) \to \mathbb{C}$ is Scott continuous iff it is weak-* continuous iff there exists $\rho \in \mathcal{DM}(\mathcal{H})$. $\phi(a) = \mathrm{tr}(\rho a)$ for all $a \in B(\mathcal{H})$. These are called *normal states*.
- Don't need to use topologies, and state-effect duality is state transformer-predicate transformer duality done using Scott continuity.

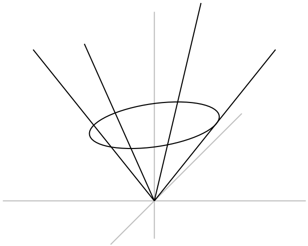
Domain Theory and Generalized Probabilistic Theories

- Does this carry over to state-effect duality for convex sets?
- If it did, we would have a way to interpret programming languages describing quantum protocols in generalized probablistic theories as well using the same concepts.
- Promising start: $\mathcal{E}(X)$ is a dcpo, and direct sets converge (weak-*) to their least upper bounds. Elements of X define Scott-continuous states on $\mathcal{E}(X)$.
- If we define $\mathcal{SCS}(A)$ to be the Scott-continuous states on A, is the evaluation map $X \to \mathcal{SCS}(\mathcal{E}(X))$ an isomorphism?
- Answer: No, not even if X is the base of a base-norm space.

The Counterexample

- Every closed bounded convex subset of a Banach space E can be made into the base of a Banach base-norm space.
- Why not use the closed unit ball of E?
- $\mathcal{BN}(E) = E \times \mathbb{R}$. The trace is the map $(x, y) \mapsto y$.
- Positive cone:

$$\begin{aligned} &\{\mathbb{R}_{\geq 0} \text{ multiples of } \mathrm{Ball}(E) \times \{1\}\} \\ =&\{(x, y) \in E \times \mathbb{R} \mid \|x\|_E \leq y\} \end{aligned}$$



The Counterexample II

- We can also make an order-unit space $\mathcal{OU}(E)$, using the same cone, and taking $(0,1)$ as the unit element.
- $\mathcal{BN}(E) \cong E \oplus_\infty \mathbb{R} = E \times \mathbb{R}$ and $\mathcal{OU}(E) \cong E \oplus_1 \mathbb{R} = E + \mathbb{R}$.
- By generalizing the isomorphisms $\ell^\infty(2)^* \cong \ell^1(2)$ and $\ell^1(2)^* \cong \ell^\infty(2)$, we get isomorphisms $\mathcal{BN}(E)^* \cong \mathcal{OU}(E^*)$ and $\mathcal{OU}(E)^* \cong \mathcal{BN}(E^*)$.
- Already at this point we can import counterexamples from Banach space theory, *e.g.* a convex set X such that $X \cong \mathcal{S}(\mathcal{E}(X))$ but the evaluation map is not an isomorphism.

The Counterexample III

- $\mathcal{BN}(E)^*$ is bounded-directed complete, because it's isomorphic to $\mathcal{E}_\pm(\mathrm{Ball}(E))$.
- By analysing it as $\mathcal{OU}(E^*)$, we see that if x is the least upper bound of $(x_i)_{i \in I}$, then $x_i \to x$ in *norm*, not just weak-*.
- Therefore every state on $\mathcal{BN}(E)^*$ is Scott continuous.
- If we take E to be any non-reflexive space, *e.g.* ℓ^1 or ℓ^∞, $X = \mathrm{Ball}(E)$ is a convex set such that the evaluation map $X \to \mathcal{SCS}(\mathcal{E}(X))$ is not an isomorphism.
- So an infinite-dimensional cubical bit $[0, 1]^{\mathbb{N}}$ is such an example.

Conclusion

- Don't take the topology away!
- There are other examples even in finite-dimensional quantum mechanics where using only order-theoretic approximation is a bad idea – 1-dimensional projections form a discrete set in $B(\mathcal{H})$ in the Scott topology, so you cannot approximate projections from each other using domain-theoretic notions.